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Some Results on LP-Sasakian Manifolds

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Abstract: The object of the present paper is to study LP-Sasakian manifolds satisfying certain conditions.

Key words: LP-Sasakian manifolds, quasi-conformal curvature tensor, pseudo projective curvature tensor.

1. Introduction

In 1989 K. Matsumoto [2] introduced the notion of LP-sasakian manifold. Then I. Mihai & R. Rosca [1] obtained several results in this manifold. Other geometers also studied it.

In 1968 Yano & Sawaki [4] defined & studied quasi-conformal curvature tensor which includes both the conformal and concircular curvature tensor. The present paper deals with a study of LP-Sasakian manifolds of dimension $(2n+1)$ satisfying certain conditions. After preliminaries, in section 3 we study an LP-Sasakian manifolds satisfying $R(X,Y).W=0$ & it is shown that such a manifold in an η- Einstein manifold. Also in such a manifold we obtain the value of scalar curvature & show that manifold is quasi-confromally flat. Section 4 deals with a ϕ-pseudo projectively flat LP-Sasakian manifold & it is proved that such a manifold is η-Einstein.

2. Preliminaries

A $(2n+1)$ dimensional differentiable manifold M is said to be an LP-Sasakian manifold [2], if admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi)=-1,\phi^2(X)=X+\eta(X)\xi,\phi\xi=0,\eta(\phi X)=0 \tag{2.1}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y), \quad g(X,\xi)=\eta(X) \tag{2.2}$$

$$\nabla_X\xi=\phi X,(\nabla_X\phi)(Y)=g(X,Y)\xi+\eta(Y)X+2\eta(X)\eta(Y)\xi \tag{2.3}$$

Where ∇ denotes the operator of covariant differentiation with respect to g.
The fundamental 2-form Ω is defined as

$$\Omega(X,Y)=g(X,\phi Y)=g(\phi X,Y) \tag{2.4}$$

for any vector fields X and Y, $\Omega(X,Y)$ is the symmetric (0,2) tensor field [2].

Also the vector field η is closed in an LP-Sasakian manifold we have [2], [3]

$$(\nabla_X\eta)(Y)=\Omega(X,Y)=g(X,\phi Y),\Omega(X,\xi)=0 \tag{2.5}$$

for any vector fields X and Y.

A $(2n+1)$-dimensional LP-Sasakian manifold is said to be an Einstein & an η- Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y)=\alpha g(X,Y)\,\&\,S(X,Y)=\beta g(X,Y)+\gamma\eta(X)\eta(Y) \tag{2.6}$$

respectively where $\alpha,\beta\,\&\,\gamma$ are smooth functions on M.

Also we have the following important basic results [2]

$$\eta(R(X,Y)Z)=g(Y,Z)\eta(X)-g(X,Z)\eta(Y) \tag{2.7}$$

$$R(\xi,X)Y=g(X,Y)\xi-\eta(Y)X \tag{2.8}$$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y \tag{2.9}$$

$$R(\xi,X)\xi=X+\eta(X)\xi=\phi^2X \tag{2.10}$$

$$S(X,\xi)=2n\eta(X) \tag{2.11}$$

$$S(\phi X,\phi Y)=S(X,Y)+2n\eta(X)\eta(Y) \tag{2.12}$$

for any vector fields X,Y,Z where R(X,Y)Z is the Riemannian curvature tensor.

The quasi-conformal curvature tensor W on a manifold M of dimension $(2n+1)$ is defined by [4]

$$W(X,Y)Z=aR(X,Y)Z+b\left[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\right]$$
$$-\frac{r}{2n+1}\left\{\frac{a}{2n}+2b\right\}\left[g(Y,Z)X-g(X,Z)Y\right] \tag{2.13}$$

Where a,b are arbitrary constants, $a,b\neq 0$ and other symbols have their usual meanings.

3. LP-Sasakian manifolds satisfying *R(X, Y)W* = 0

Let us consider an LP-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition [6]

$$R(X,Y).W=0 \tag{3.1}$$

Now,

$$\begin{aligned}(R(X,Y).W)(U,V)Z&=R(X,Y)W(U,V)Z-W(R(X,Y)U,V)Z\\&-W(U,R(X,Y)V)Z-W(U,V)R(X,Y)Z\end{aligned} \tag{3.2}$$

From relations (3.1) and (3.2) we have

$$R(X,Y)W(U,V)Z-W(R(X,Y)U,V)Z-W(U,R(X,Y)V)Z-W(U,V)R(X,Y)Z=0 \tag{3.3}$$

Taking $X=\xi$ in (3.3) we obtain by virtue of (2.8)

$$\begin{aligned}&g(W(U,V)Z,Y)\xi-\eta(W(U,V)Z)Y-g(Y,U)W(\xi,V)Z+\eta(U)W(Y,V)Z\\&-g(Y,V)W(U,\xi)Z+\eta(V)W(U,Y)Z-g(Y,Z)W(U,V)\xi+\eta(Z)W(U,V)Y=0\end{aligned} \tag{3.4}$$

Taking innerproduct on both sides by ξ in (3.4) & using $\eta(W(U,V)\xi)=0$, we get

$$-g(W(U,V)Z,Y)-\eta(Y)\eta(W(U,V)Z)-g(Y,U)\eta(W(\xi,V)Z)+\eta(U)\eta(W(Y,V)Z)$$
$$-g(Y,V)\eta(W(U,\xi)Z)+\eta(V)\eta(W(U,Y)Z)+\eta(Z)\eta(W(U,V)Y)=0 \tag{3.5}$$

Let $\{e_i : i=1,2,......,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then setting $U=Y=e_i$ and taking summation over $i, 1\le i\le 2n+1$ in (3.5) we get

$$-\sum_{i=1}^{2n+1} g(W(e_i,V)Z,e_i)-(2n+1)\eta(W(\xi,V)Z)-\sum_{i=1}^{2n+1} g(e_i,V)\eta(W(e_i,\xi)Z)+$$
$$\eta(Z)\sum_{i=1}^{2n+1}\eta(W(e_i,V)e_i)=0 \tag{3.6}$$

Since, $\sum_{i=1}^{2n+1}\eta(V)\eta(W(e_i,e_i)Z)=0$

After straight forward calculation we get

$$S(V,Z)=\frac{1}{a-b}\left[2n\{a+2nb\}-br\right]g(V,Z)+\frac{b}{a-b}\left[2n(2n+1)-r\right]\eta(V)\eta(Z) \tag{3.7}$$

Provided that $a-b\neq 0$

Hence we can state

Theorem 3.1 An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W=0$ is an η-Einstein manifold, providing $a-b\neq 0$.

Let $\{e_i : i=1,2,......,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $V=Z=e_i$ in relation (3.7) & then summing over $i, 1\le i\le 2n+1$, we get $r=2n(2n+1)$, if $a-b\neq 0$, (3.8)

This leads to the corollary:

Corollary 3.1: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W=0$ has scalar curvature given by (3.8).

Using (3.8) in the relation (3.7) we get

$$S(V,Z)=2ng(V,Z), \quad a-b\neq 0, \tag{3.9}$$

Thus we can state the following theorem:

Theorem3.2: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W=0$ is an Einstein manifold providing $a-b\neq 0$.

Putting $Z=\xi$ in relation (3.9) we get

$$S(V,\xi)=2n\eta(V) \tag{3.10}$$

Now, taking inner product on both sides of (2.13) by ξ & using the relations (3.8), (3.9), (3.10) and (2.7) we obtain

$$\eta(W(U,V)Z)=0 \qquad \text{for all } U,V,Z \tag{3.11}$$

Again, using (3.11) in the relation (3.5) we get

$$g(W(U,V)Z,Y)=0 \qquad \text{for all } U,V,Z$$

This implies

$$W(U,V)Z=0 \qquad \text{for all } U,V,Z \tag{3.12}$$

Hence we can state

Theorem 3.3: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W=0$ is quasi-conformally flat.

4. ϕ-pseudo Projectively flat LP- Sasakian Manifold

Definition: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be ϕ-pseudo projectively flat if it satisfies

$$\phi^2(\bar{P}(\phi X,\phi Y)\phi Z)=0 \tag{4.1}$$

for arbitrary vector fields $X,Y,Z\in T_pM$.

The pseudo projective curvature tensor is defined by [5]

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y\right] - \frac{r}{2n+1}\left\{\frac{a}{2n}+b\right\}\left[g(Y,Z)X - g(X,Z)Y\right] \tag{4.2}$$

Where a,b are constants such that $a,b \neq 0$, R,S and r are the curvature tensor, Ricci-tensor & scalar curvature respectively.

Consider an LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ which is ϕ-pseudo projectively flat then (4.1) holds. Now from relations (4.1) & (4.2) we have

$$g\left(\overline{P}(\phi X,\phi Y)\phi Z,\phi W\right) = ag\left(R(\phi X,\phi Y)\phi Z,\phi W\right) + b\left[S(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W)\right]$$
$$-\frac{r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}\left[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\right]$$

or

$$\tilde{R}(\phi X,\phi Y,\phi Z,\phi W) = -\frac{b}{a}\left[S(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W)\right]$$
$$+\frac{r}{(2n+1)a}\left\{\frac{a}{2n}+b\right\}\left[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\right] \tag{4.3}$$

Where $\tilde{R}(\phi X,\phi Y,\phi Z,\phi W) = g\left(R(\phi X,\phi Y)\phi Z,\phi W\right)$.

Using, (2.2) and (2.12) in (4.3) we get

$$g(R(X,Y)Z,W) + g(Y,Z)\eta(X)\eta(W) + g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)\eta(W) - g(Y,W)\eta(X)\eta(Z)$$
$$= \frac{b}{a}\left[-\{S(Y,Z)+2n\eta(Y)\eta(Z)\}\{g(X,W)+\eta(X)\eta(W)\} + \{S(X,Z)+2n\eta(X)\eta(Z)\}\{g(Y,W)+\eta(Y)\eta(W)\}\right]$$
$$+\frac{r}{(2n+1)a}\left\{\frac{a}{2n}+b\right\}\times$$
$$\left[\{g(Y,Z)+\eta(Y)\eta(Z)\}\{g(X,W)+\eta(X)\eta(W)\} - \{g(X,Z)+\eta(X)\eta(Y)\}\{g(Y,W)+\eta(Y)\eta(W)\}\right] \tag{4.4}$$

Where $g\left(R(X,Y)Z,W\right) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W)$.

Let $\{e_i : i = 1,2,......,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = W = e_i$ in (4.4) & taking sum over $i, 1 \leq i \leq 2n+1$, we get

$$S(Y,Z)=\left[\frac{1}{a+(2n+1)b}\left\{\frac{(a+2nb)r}{2n}-a\right\}\right]g(Y,Z)+\left[\frac{(a+2nb)(2n-1)}{a+(2n+1)b}\left\{\frac{r}{2n(2n+1)}-1\right\}\right]\eta(Y)\eta(Z) \tag{4.5}$$

Provided that $a,b\neq 0$.

This leads to the following:

Theorem 4.1: A ϕ-pseudo projectively flat LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is an η-Einstein manifold, providing $a,b\neq 0$.

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