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Structure of Ultradistribution

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Abstract: Structure of ultradistribution belonging to $\left(H_{\mu,a_k,A}^{b_q,B}\right)$ has been investigated

1. Introduction:

The classical Hankel transformation is defined by

$$(1.1) \qquad (h_\mu \phi)(x) = \int_0^\infty \phi(y)\sqrt{xy J_\mu(xy)dy}, \quad \left(\mu = -\frac{1}{2}\right)$$

where J_μ is the Bessel function of the first kind and order μ. Zemanian [5] introduced the spaces H'_μ and its dual H'_μ to extend the above transformation to the space of distributions belonging to H'_μ. Following the technique of Gel'fand and Shilov [1], Lee [2] defined spaces $H_{\mu,\alpha,A}, H_\mu^{\beta,B}$ and $H_{\mu,\alpha,A}^{\beta,B}$, Pathak and Pandey [3] introduced spaces $H_{\mu,a_k,A}$, $H_\mu^{b_q,B}$ and $\left(H_{\mu,a_k,A}^{b_q,B}\right)$ generalizing the aforesaid Lee spaces. The dual spaces of $H_{\mu,a_k,A}$, $H_\mu^{\beta_q,B}$ and $\left(H_{\mu,a_k,A}^{b_q,B}\right)$ are $(H_{\mu,a_k,A})'$, $(H_\mu^{b_q,B})'$ and $\left(H_{\mu,a_k,A}^{b_q,B}\right)$. The elements of the dual spaces are called ultradistributions. In the present work, we study the structure of ultradistribution.

Throughout the paper, I denote the open interval $(0, \infty)$ and all the testing functions herein are defined on I. We recall here the spaces $H_{\mu,a_k,A}$, $H_\mu^{b_q,B}$ and $H_{\mu,a_k,A}^{b_q,B}$ defined by Pathak and Pandey.

Let $\{a_k\}_{k\in IN}$ and $\{b_q\}_{q\in IN}$ be arbitrary sequence of positive numbers which satisfy the following conditions:

(i) Logarithmic Convexity

$$(1.2) \qquad a_k^2 \le a_{k-1}\, a_{k+1}, \qquad k \ge 1$$

$$(1.3) \qquad a_q^2 \le b_{q-1}\, a_{q+1}, \qquad q \ge 1$$

Immediate consequences of these inequalities are

(1.4) $a_p a_k \le a_0 a_{p+k};\qquad p, k = 0,1,2$

(1.5) $b_p b_q \le b_0 a_{p+q};\qquad p, q = 0,1,2$

(ii) Stability under multiplication by x

There are constants c, h, c_1 and h_1 such that $k \ge 0,\ q \ge 0$

(1.6) $a_{k+1} \le ch^k\, a_k$

(1.7) $b_{q+1} \le c_1 h_1^q\, b_q$

(iii) Non-quasi analyticity

(1.8) $$\sum_{q=1}^{\infty} \frac{b_{q-1}}{b_q} < \infty$$

(iv) Stability under Hankel transformations

Conditions (1.6) and (1.7) are replaced by the following stronger conditions:

(1.9) $a_{r+k} \le LR^{r+k}\, a_r a_k \quad$ for all $r, k \ge 0$

(1.10) $b_{r+q} \le R_1^{r+q}\, b_r b_q \quad$ for all $r, q \ge 0$

where L, R, L_1 and R_1 are positive constants.

Pathak and Pandey have introduced spaces $H_{\mu,a_k,A}$, $H_{\mu}^{b_q,B}$ and $H_{\mu,a_k,A}^{b_q,B}$.

These spaces are defines as follows :

Let ϕ be an infinitely differentiable function on $I = (0, \infty)$. Then

(a) $\phi \in H_{\mu,a_k,A}$ if and only if

$$\gamma_{k,q}^{\mu}(\phi) = \sup_{x\in I} \left| x^k (x^{-1}D^q(x^{-\mu-1/2})\phi(x) \right| \le C_q^{\mu}(A+\delta)^k a_k;\qquad k, q \in IN_0$$

where the constants A and C_q^{μ} depend on ϕ and $\delta > 0$ is an arbitrary constant

(b) $\phi \in H_{\mu}^{b_q,B}$ if and only if

$$\gamma_{k,q}^{\mu}(\phi) = \sup_{x\in I} \left| x^k (x^{-1}D^q(x^{-\mu-1/2})\phi(x) \right| \le C_k^{\mu}(B+\rho)^q bq,\qquad k, q \in IN_0$$

where the constant C_q^{μ} depend on ϕ and $\rho > 0$ is arbitrary constant.

(c) $\phi \in H_{\mu,a_k,A}^{b_q,B}$ if and only if

$$\gamma_{k,q}^{\mu}(\phi) = \sup_{x\in I} \left| x^k (x^{-1}D)^q (x^{-\mu-1/2})\phi(x) \right|$$

$$\le C^{\mu}(A+\delta)^k (B+\rho)^q a_k b_q,\qquad k, q \in IN_0$$

where δ and ρ are as above and c^{μ}, A and B are certain positive constants depending on ϕ.

2. Structure of ultradistribution

In this section we shall investigate the structure of ultradistribution belonging to $\left(H^{b_q,B}_{\mu,a_k,A}\right)'$.

Theorem 2.1 *A linear functional u defined on* $H^{b_q,B}_{\mu,a_k,A}$ *belongs to* $\left(H^{b_q,B}_{\mu,a_k,A}\right)'$ *if and only if there exist* $r \in IN$ *and function* $f_p \in L^{\infty}(I)$ $(0 \le p \le r)$ *such that*

$$u = \sum_{p=0}^{r} x^{-p-1/2}\left(-D\frac{I}{x}\right)^{q} x^{k} f_p.$$

Proof: Let $u \in \left(H^{b_q,B}_{\mu,a_k,A}\right)'$. Then there exists a positive constant c and a non-negative integer r such that for every $\phi \in H^{b_q,B}_{\mu,a_k,A}$.

$$(2.1) \qquad |< u,\phi >| = c \max_{\substack{0\le k\le r \\ 0\le k\le r}} \sup_{x\in I} \left| x^{k}(x^{-1}D)^{q} x^{-\mu-1/2}\phi(x)\right|$$

Let Γ denote the direct sum of $(r+1)$ copies of $L'(I)$ normed with

$$\left|(f_j)_{0\le j\le r}\right|_1 = \max_{0\le j\le r}\|f_j\|_1$$

and β be the denote the direct sum of $(r+1)$ copies of $L^{\infty}(I)$ normed with

$$\left|(f_j)_{0\le j\le r}\right|_{\infty} = \sum_{j=0}^{r}\|f_j\|_{\infty}$$

Now consider the mapping

$$F : H^{b_q,B}_{\mu,a_k,A} \to \Gamma$$

$$\phi \to F(\phi) = \left(x^{k}(x^{-1}D)^{q}\ x^{-\mu-1/2}\ \phi(x)\right)_{0\le q\le r}$$

Clearly the mapping is one-one.

Define the functional L on $F\left(H^{b_q,B}_{\mu,a_k,A}\right) \subset \Gamma$ by

$$< L, F(\phi) > = < u, \phi > \text{ for } \phi \in H^{b_q,B}_{\mu,a_k,A}$$

L is continuous from F $\left(H^{b_q,B}_{\mu,a_k,A}\right)$ into C by virtue of (1), when $F\left(H^{b_q,B}_{\mu,a_k,A}\right)$ is equipped with the topology induce by Γ. By Hahn- Banach's theorem, we can extend continuously upto Γ without increasing the norm. This extension is also denoted by L. Since $L \in \Gamma'$, therefore for $(g_p)_{p=0}^{r} \in \Gamma$ and for certain $f_p \in L^{\infty}$, Reisz's representation theorem gives

$$< L, (g_p)_{p=0}^{r} > = \sum_{p=0}^{r}\int_{0}^{\infty} f_p(x) g_p(x) dx$$

Now,

$$< u, \phi > = < L, F(\phi) >$$

$$= \sum_{p=0}^{r} \int_0^\infty f_p(x)\, x^k (x^{-1}D)^q\, x^{-\mu-1/2}\, \phi(x)dx$$

$$= < \sum_{p=0}^{r} x^{-\mu-1/2} \left(-D\frac{1}{x}\right)^q x^k f_p, \phi >$$

$$\therefore u = \sum_{p=0}^{r} x^{-\mu-1/2} \left(-D\frac{1}{x}\right)^q x^k f_p,$$

Conversely, let

$$< u, \phi > = \sum_{p=0}^{r} \int_0^\infty f_p(x) x^k (x^{-1}D)^q x^{-\mu-1/2}\, \phi(x)dx, \left(\phi(x) \in H_{\mu,a_k,A}^{b_q,B}\right)$$

Then

$$|< u, \phi >| = \sum_{p=0}^{r} \int_0^\infty \left| f_p(x) x^k (x^{-1}D)^q x^{-\mu-1/2} \phi(x) \right| dx$$

$$\le \sum_{p=0}^{r} \|f_p\| \int_0^\infty \left| x^k (x^{-1}D)^q\, x^{-\mu-1/2}\, \phi(x) \right| dx$$

Now,

$$\int_0^\infty \left| x^k (x^{-1}D)^q\, x^{-\mu-1/2}\, \phi(x) \right| dx$$

$$= \int_0^{-1} \left| x^k (x^{-1}D)^q\, x^{-\mu-1/2}\, \phi(x) \right| dx + \int_0^\infty \left| x^k (x^{-1}D)^q x^{-\mu-1/2} \phi(x) \right| dx$$

$$\le \sup_x \int_0^\infty \left| x^k (x^{-1}D)^q x^{-\mu-1/2}\, \phi(x) \right| \sup_x \int_0^1 \left| x^{k+2} (x^{-1}D)^q x^{-\mu-1/2}\, \phi(x) \right| \int_1^\infty \frac{1}{x^2} dx$$

$$\le c^\mu (A+\delta)^k (B+\rho)^q a_k b_q + c^\mu (A+\delta)^{k+2} (B+\rho)^q a_{k+2} b_q$$

$$\le c^\mu (A+\delta)^k (B+\rho)^q a_k b_q + c^\mu (A+\delta)^{k+2} (B+\rho)^q b_q L R^{k+2} a_k a_2$$

$$= c^\mu (A+\delta)^k (B+\rho)^q a_k b_q [1 + (A+\delta)^2 a_2 L R^{k+2}]$$

$$= c_1^\mu (A+\delta)^k (B+\rho)^q a_k b_q$$

So that

$$|\langle u, \phi \rangle| = c_1^\mu (A+\delta)^k (B+\rho)^q a_k b_q \sum_{p=0}^{r} \|f_p\|_\infty$$

Hence $u \in \left(H_{\mu,a_k,A}^{b_q,B}\right)'$.

This completes the proof of the theorem.

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