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Study of Some Property of Density Topology

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Abstract: In this paper we prove that in the plane density topology exists without power property.

Key words: Discrete topology, density topology, power property.

Introduction

Let X be a topological space. Any continuous transformation, on a topological space X will be called mapping. We say that $f : A \to X$ is light on $A \subset X$ if f is non-constant on any non-degenerate continuous mapping in A.

We say that any topology on X has the power property if for any open set $U \subset X$, any open and light mapping $f : U \to X$ and any point $x \in U$ there exist a number $n \in N$ and a *nbd* $V \subset U$ of x such that for every $y \in [(f(V)\setminus (f\{x\})]$ the set $f^{-1}(\{y\})$ has cordiality n. In order way the power property of a topology means that each open and light mapping on any open set is locally n to one.

Results

1. Family Lemma: For any $t \in (0,1)$, we define the family, $Ht = \{\frac{1}{2^k} : k \geq 0\}$. Given prison set $M \subset (0,1)$ of the lebesque measure $\lambda(m) = \frac{1}{2^n} \exists\ t \in (0,1)$ with a free family $Ft = Ht / M$ of cardiality at least n.

Proof : The assertion is obviously fulfilled with $M = \left(0, \frac{1}{2^n}\right)$. We can try to avoid the free family of cordiality at least $(n+1)$ with a prison of measure $\frac{1}{2^n}$. For each $t \in (0.5,1)$ there must be at most n members of the family $Ht = \{t / 2^k : K > 0\}$ free. The most efficient way is to build the prison $m = \left(0, \frac{1}{2^n}\right)$

Hence the family lemma proved.

2. Theorem : *The density topology does not have the power property in the plane.*

Proof : Let $D = \{r\,(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : 0 \le r < 1,\ 0 < t < 2\}$ and define a corkscrew type mapping $f : D \to C$ by $f(r(\cos \pi t + i \sin \pi t) = r\{\mathrm{Cos}\ 2\pi\phi(t) + i \sin 2\ \pi\phi(t)\}$. When,

$$\phi(t) = \begin{cases} t-1, \text{for } t \in (1,2] \\ \phi(2^{n+1}t), \text{for } t \in \left(\frac{1}{2^{n+1}} \cdot \frac{1}{2^n}\right], \end{cases}$$

$n \ge 0$ with t decreasing from 2 to 1 the lower half of D is mapped onto D Anti-clockwise, then the speed of rotation increases in such a way that,

$$f[\{r(\cos \pi t + i \sin \pi t) \in C : 0 \le r\ \ 1, t \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)\}] = D \text{ for } n \ge 0.$$

We can consider the mapping $f : D \to C$ with the density topology on both D and C. We have

A) D is a density open set (the missing segment has the lebesgues measure zero).

B) f is density continuous at $D \setminus \{0\}$.

C) f is density continuous at 0.

Proof of (C) : For any density open set V containing 0, the density of a set $f^{-1}(V)$ at 0 can be calculated using the radical segments.

$[\{r(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : \mathrm{O} \le r < 1,\ t \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)\}]$ of D defined by the segments $t \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$, the density of V gives the density of $f^{-1}(V)$ at 0.

D) f is density open at $D\setminus \{0\}$.

E) f is density open at $\{0\}$

Proof of (E) : The density of U at o gives the estimate of the lebesgue measure of

$$U \cap \{r\,(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : \mathrm{O} \le r < 1, t \in (1,2)\}$$

And we obtain the estimate of the density of $f(U)$ at 0.

F) f is a light and open mapping on an open set D on a topological space C with the density topology.

G) For any density open set $V \subset D$ containing 0 and $n \in N$ there exists $y \in f(V)$ such that the set $V \cap f^{-1}(y)$ has cardinality at least n.

Proof of (G) : There is a density open set $U \subset V$, containing 0 and a open set G (Euclidean open) containing the density closed set $C\backslash V$, such that G and U are disjoint (See–[2]), the Lusin–Menchoff property of the density topology. When U reaches the density $\left(1 - \frac{1}{2^n}\right)$ at 0 for some $R \in (0, 1)$, i.e., $\lambda\,[\{r\,(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : 0 \le r < r, t \in (0, 2)\}] > \left(1 - \frac{1}{2^n}\right).\ \pi R^2$. We can using the polar coordinates obtain $r \in (0, R)$ such that the set,

$$M = \{t \in (0, 2) : r(\cos \pi t + i \sin \pi t) \in G\}$$

is Euclidean open in (0,2) with $\lambda(M) \leq \frac{1}{2^n}$.

By family lemma with M as the prison set we conclude that there exists $t \in (0,1)$ with the free family set $Ft = \{t_1, \dots t_n\} \subset (0,1)$ disjoint with M, being of cardinality at least n.

Then $y = f\{r(\cos \pi t_i + i \sin \pi t_i)\}$
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$= f\{r(\text{Cos}\, \pi t_n + i \sin \pi t_n) \in f(V)$ due to the definitions of Ft and f, and consequently $f^{-1}(y)$ has cardinality at least n.

The mapping $f : D \to C$ shows that the density topology does not have the power property. Hence the theorem

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REFERENCE

[1] G.T. Whyburn, *Topological Analysis*, Princeton University press, Princeton 1958.

[2] J.Lukeš, J. Maly, L.Zajicek, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Mathematics 1189, springer–verlag, Berlin 1986.

[3] S.Stölow, *Principles Topologiques de la Theoric des Fonctions analytiques*, Paris, Gauthier–Villars 1938.

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