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Submanifold of Codimension p of a HSU-Structure Manifold

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Abstract : Hsu-structure manifolds have been defined and studied by Prof. Mishra [2]. Islam and others. The purpose of the present paper is to study the submanifolds of such a manifold. It has been shown that a submanifold of condimension p of such a manifold admits a para p-contact Hsu-metric structure. Certain other interesting results have also been proved.

1. Preliminaries

Let V_n be an n-dimensional differentiable manifold of class C^∞. Suppose there exists on V_n a tensor field $F(\neq 0)$ of type (1,1) satisfying

$$F^2 = a^r I \tag{1.1}$$

where 'a' is any non-zero complex number and r a positive integer. Suppose further that V_n admits a hermite metric G satisfying

$$G(FX, FY) + a^r G(X,Y) = 0 \tag{1.2}$$

for arbitrary vector fields X and Y on V_n. Thus, in view of the equations (1.1) and (1.2) V_n will be said to possess a Hsu-metric structure.

Let $'F(X,Y)$ is the tensor field of type (1.2) given by

$$'F(X,Y) = G(FX,Y). \tag{1.3}$$

The following results can be proved easily

(1.4)

(i) $\quad 'F(FX,Y) = -'F(X,FY) = a^r G(X,Y)$

(ii) $\quad 'F(FX,FY) + a^r\, 'F(X,Y) = 0$ and

$'F(X,Y) + 'F(Y,X) = 0$

Let $\overline{D}$ be the Riemannian connection on V_n ; then

(1.5) $$\overline{D}_X Y - \overline{D}_Y Y = [X,Y] \qquad \overline{D}_X G = 0$$

Let $\overline{N}$ be the Nijenhuis tensor formed with F ; then

(1.6) $$\overline{N}(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2 [X,Y]$$

A Hsu-structure manifold V_n will be called a HK-manifold if the structure tensor F is parallel i.e.

(1.7) $$(\overline{D}_X F)(Y) = 0.$$

A sub manifold V_{n-p} of condimesion p of the Hsu-structure manifold V_n will be said to possess a para p-contact Hsu-structure if there exists a tensor field f of type (1,1) p(c*) contravariant vector fields U p(c*) 1-forms $\overset{X}{u}$ (p some finite integer) satisfying

(1.8) $$f^2 = a^r I - \sum_{X=1}^{p} \overset{X}{u} \otimes \underset{X}{U}$$

Also,

$$\overset{Y}{u} \circ f + \sum_{X=1}^{p} \theta_X^Y \overset{X}{u} = 0 \qquad f \underset{X}{U} + \sum_{Y=1}^{p} \theta_X^Y \underset{X}{U} = 0$$

(1.9) $$\overset{Y}{u}(\underset{Y}{U}) + \sum_{Z=1}^{p} \theta_Z^X \theta_Y^Z = a^r \delta_Y^X$$

where X, Y = 1, 2, . . . p, δ_Y^X denotes the Kronecker delta and δ_Y^X are scalar fields.

If in addition, the submanifold V_{n-p} admits a Riemannian metric g satisfying

(1.10) $$g(fX, fY) + a^r g(X,Y) + \sum_{X=1}^{p} \overset{X}{u}(X) \overset{X}{u}(Y) = 0$$

we say that V_{n-p} admits a para p-contact Hsu-metric structure.

2. Submanifolds of Codimension p

Let V_{n-p} be the submanifold of codimension p of a Hsu-structure menifold V_n If B denotes the differential of the immersion $\tau : V_{n-p} \rightarrow V_n$ a vector field X in the tangent space of V_{n-p} determines a vector field BX in that of V_n . Let $\underset{X}{N}$, $X = 1,2, \ldots, p$ be p mutually orthogonal fields of unit normal vectors defined on V_{n-p}. Thus, we have

(2.1) $G(BX,BY) = g(X,Y) \quad G(BX,N) = 0 \quad G(\underset{X}{N}, \underset{Y}{N}) = \delta_Y^X$

The vector fields FBX and $F\underset{X}{N}$ can be expressed by

$$\text{(i)} \quad FBX = B\,fX - \sum_{X=1}^{p} \overset{X}{u}(X)\underset{X}{N}$$

(2.2)

$$\text{(ii)} \quad F\underset{X}{N} = -B\underset{X}{U} + \sum_{Y=1}^{p} \theta_X^Y \underset{X}{N}$$

where f is a (1,1) tensor field, $\overset{X}{u}$ 1-forms and $\underset{X}{U}$ vector fields on the submanifold V_{n-p}. Operating by F on both the sides of (2.2)(i) and making use of equation (1,1) and (2.2), we obtain

$$a^r BX = Bf^2X - \sum_{Y=1}^{p} \overset{Y}{u}(fX)\underset{Y}{N} - \sum_{X=1}^{p} \overset{X}{u}(X)\left\{-B\underset{x}{U} + \sum_{Y=1}^{p} \theta_x^y \underset{y}{N}\right\}.$$

Comparison of tangential and normal vectors gives

(2.3) $$f^2 = a^r I - \sum_{X=1}^{p} \overset{Z}{u} \otimes \underset{X}{U} \qquad \overset{Y}{u} \circ f + \sum_{X=1}^{p} \theta_X^Y \overset{X}{u} = 0$$

Multiplying both the sides of equation (2.2)(ii) by F and using again equation (1.1) and (2.2), we get

$$a^r \underset{X}{N} = \left\{-Bf\underset{X}{U} - \sum_{Z=1}^{p} \overset{X}{u}(\underset{X}{U})\underset{Y}{N}\right\} + \sum_{Y=1}^{p} \theta_X^Y \left\{-B\underset{Y}{U} + \sum_{Z=1}^{p} \theta_Y^Z \underset{Z}{N}\right\}$$

Comparison of tangential and normal vectors gives

(2.4) $$f\underset{Z}{U} + \sum_{Y=1}^{p} \theta_X^Y \underset{Y}{U} = 0 \qquad \overset{X}{u}(\underset{X}{U}) + \sum_{Y=1}^{p} \theta_Y^Z \theta_X^Y = a^r \delta_X^Z$$

Further in view of the equations (1.1), (2.1) and (2.2), if g is the induced metric on V_{n-p} then we have

(2.5) $$g(fX, fY) + a^r g(X,Y) + \sum_{X=1}^{p} \overset{X}{u}(X)\,\overset{X}{u}(Y) = 0$$

In view of the equation (2.3), (2.4) and(2.5), we have

Theorem 2.1. *The submanifold* V_{n-p} *of codimension p of a Hsu-structure manifold* V_n *admits a para p-contact Hsu-metric structure.*

Suppose further that $\overline{D}$ is the Riemannian connection on V_n and D the induced connection on the submanifold V_{n-p}. Then the equations of Gauss and Weingarten can be expressed as

$$\text{(2.6)} \qquad \overline{D}_{BX} BY = BD_X Y + \sum_{X=1}^{p} \overset{X}{h}(X,Y) \underset{X}{N}$$

$$\text{(2.7)} \qquad \overline{D}_{BX} \underset{X}{N} = -B\overset{X}{H}(X) + \sum_{Y=1}^{p} \theta_X^Y \underset{Y}{N}$$

where $\overset{X}{h}(X,Y)$ are second fundamental forms, and

$$\text{(2.8)} \qquad \overset{X}{h}(X,Y) = g(\overset{X}{H}(X),Y).$$

Suppose that the enveloping manifold V_n is a HK-manifold. Hence we have $(\overline{D}_{BX} F)(BY) = 0$ or equivalently

$$\overline{D}_{BX} FBY = F\overline{D}_{BX} BY.$$

In view of the equations (2.2), (2.6) and (2.7), the last equation takes the form

$$D_{BX} = \left\{ BfY - \sum_{X=1}^{p} \overset{X}{u}(Y)\underset{X}{N} \right\} = F\left\{ BD_X Y + \sum_{X=1}^{p} \overset{X}{h}(X,Y)\underset{Y}{N} \right\}$$

or equivalently

$$BD_X fY + \sum_{X=1}^{p} \overset{X}{h}(X,fY)\underset{X}{N} - \sum_{X=1}^{p} \overset{X}{u}(Y)\left\{ -B\overset{X}{H}(X) + \sum_{Z=1}^{p} \theta_X^Y \underset{Y}{N} \right\}$$

$$= BD_x Y - \sum_{X=1}^{p} \overset{X}{u}(D_x Y)\underset{X}{N} + \sum_{X=1}^{p} \overset{X}{h}(X,Y)\left\{ -B\underset{X}{U} + \sum_{Y=1}^{p} \theta_X^Y \underset{Y}{N} \right\}$$

The comparison of the tangential vectors gives

$$D_X Y + \sum_{X=1}^{p} \overset{X}{u}(Y)\overset{X}{H}(X) = fD_X Y - \sum_{X=1}^{p} \overset{X}{h}(X,Y)\underset{X}{U}$$

or equivalently

$$\text{(2.9)} \qquad (D_x f)(Y) + \sum_{X=1}^{p} + \left\{ \overset{X}{u}(Y)\overset{X}{H}(X) + \overset{X}{u}(X,Y)\underset{X}{U} \right\} = 0$$

If N(X,Y) is the Nijenhuis tensor for the submanifold V_{n-p} we can write

$$N(X,Y) = (D_{fX} f)(Y)\ (D_{fY} f)(X) + f(D_Y f)(X) - f(D_X f)(Y)$$

A necessary and sufficient condition that the submanifold V_{n-p} be totally geodesic is that $\overset{x}{h}(X,Y) = 0$ $(X = 1,2,\ldots p)$. Thus, in view of the equations (2.8) and (2.9) it follows that $D_X f = 0$. Hence from (2.1) we have $N(X,Y) = 0$.

But V_{n-p} is said to be integrable if and only if $N(X,Y) = 0$. Thus, we have

Theorem 2.2. *A totally geodesic submanifold* V_{n-p} *with a para p-contact Hsu-structure of a Hsu-structure manifold is integrable*

3. Curvature Tensor

Suppose that W, X, Y, Z are arbitrary vector fields on an open set A in the neighbourhood of a point of the sub manifold V_{n-p}. If $\overline{L}$ and L are the Riemann Chrostoffel curvature tensors of V_n and V_{n-p} respectively, we have

$$(3.1)\qquad \overline{L}(BW, BX, BY, BZ) = L(W,X,Y,Z) + \sum_{X=1}^{p} \{\overset{X}{h}(X,Z)\overset{X}{h}(W,Y) - \overset{X}{h}(X,Y)\overset{X}{h}(W,Z)\}.$$

If the manifold V_n admits constant holomorphic sectional curvature C, we have

$$(3.2)\qquad \overline{L}(BW, BX, BY, BZ) = \frac{C}{4}[G(BW, BZ)G(BX,BY) - G(BX, BZ)G(BW,BY) + {}'F(BX,BZ)\,{}'F(BX,BY) - {}'F(BX,BY)\,{}'F(BW,BZ) + 2\,{}'F(BW,BX)\,{}'F(BY,BZ)].$$

From equation (1.3) and (2.2), it can be proved that

$${}'F(BX,BY) = f(X,Y) \underset{=\!=}{def}\ g(fX,Y)$$

Hence in view of the equations (2.1), (3.1) and (3.3) the equations (3.2) takes the

$$(3.4)\qquad L(W,X,Y,Z) = \frac{C}{4}[g(W,Z)g(X,Y) - g(X,Z)g(W,Y) + {}'f(X,Z)\,{}'f(W,Y) - {}'f(X,Y)\,{}'f(W,Z) + 2\,{}'f(W,X)\,{}'f(Y,Z) + \sum_{X=1}^{p}\{\overset{X}{h}(X,Y)\overset{X}{h}(W,Z) - \overset{X}{h}(X,Z)\overset{X}{h}(W,Y)\}]$$

Thus, we have

Theorem 3.1. Let V_n *be an Hsu-structure manifold of constant holomorphic sectional curvature* C. *Then the curvature tensor of the submanifold* V_{n-p} *satisfies the equation* (3.4).

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