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Summability of Legendre Series by Uniform Lower Triangular Matrix Method

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Abstract: In this paper a new theorem on the uniform matrix summability of Legendre series has been established. This theorem is generalization of previously known all theorems of this direction.

Key words: Matrix Summability, Legendre series, orthogonal polynomial, monotonic function.
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1. Definitions

The Legendre series associated with the Lebesgue- integrable function of f (x) in the interval (-1, 1) is given by

$$f(x)=\sum_{n=0}^{\infty} a_n P_n(x) \tag{1}$$

where $a_n=(n+\frac{1}{2})\int_{-1}^{1} f(t)\, P_n(t)\, dt$ (2)

and the Legendre polynomials $P_n(x)$, which are orthogonal in the interval (−1 ,1) are defined by the generating function

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty} P_n(x)\, z^n . \tag{3}$$

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman- Töeplitz [4] conditions of regularity i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1 \text{ as } n \to \infty,$$

$$a_{n,k} = 0 \text{ , for } k > n$$

and

$$\sum_{k=0}^{n} |a_{n,k}| \le M, \text{ a finite constant.}$$

Let $\sum_{m=0}^{\infty} u_m(x)$ be infinite series defined in $a \le x \le b$.

Write

$$S_n(x) = \sum_{\nu=0}^{n} u_\nu(x).$$

If there exist a function S(x) such that

$$\sum_{k=0}^{n} a_{n,k} \left(S_k(x) - S(x)\right) = o(1) \text{ as } n \to \infty \tag{4}$$

Uniformly in set E in which S(x) is bounded, then the series $\sum_{m=0}^{\infty} u_m(x)$ is summable by matrix means (T) uniformly in set E to sum S(x).

We use the following notations.

$$N_n(t) = \sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin \frac{t}{2}}$$

$$\psi(t) = \psi(\theta, t) = f\{\cos(\theta - t)\} - f(\cos\theta)$$

$$\Psi(t) = \int_0^t |\psi(u)| \, du$$

2. Introduction

Tripathi [5], Prasad & Tripathi [2] and Prasad [3] have studied Legendre series by ordinary Nörlund summability methods as well as uniform Nörlund summability methods. The objective of this paper is to obtain a more general result than those of Tripathi [5], Prasad & Tripathi [2] and Prasad [3] so that their results come out as particular cases.

Tripathi [5] proved the following theorem on Nörlund summability of Legendre series:

Theorem 1: If

$$\int_0^t |f(x \pm u) - f(x)| du = o\left(\frac{p_\tau}{P_\tau}\right) \text{ as } n \to \infty, \qquad (5)$$

then Legendre series (1) is summable (N,p_n) to f(x) at an internal point x of the interval $(-1+\varepsilon, 1-\varepsilon)$, $\varepsilon > 0$, where $\{p_n\}$ is a real non negative and monotonic non increasing sequence such that $P_n \to \infty$ as n tends to infinite.

Prasad [3] generalized above theorem for uniform Nörlund summability in the following form:

Theorem 2: If $\alpha(t)$ denotes a function of t, $\alpha(t)$ and $\frac{t}{\alpha(t)}$ ultimately increase steadily with t,

$$\int_0^t |f(x \pm u) - f(x)| du = o\left(\frac{t}{\alpha(P_\tau)}\right) \qquad (6)$$

uniformly in a set E defined in the interval (–1, 1), in which f(x) is bounded as $t \to +0$ then the series (1) is summable (N, p_n) uniformly in E to the sum f(x), where $\{p_n\}$ is real non negative and monotonic non increasing sequence such that $P_n \to \infty$ as $n \to \infty$, provided that

$$\log n = O(\alpha(P_n)) \text{as } n \to \infty \text{ holds.}$$

Main theorem:

If $$\int_t^\eta \frac{|\psi(u)| A_{n,\left(\frac{1}{u}\right)}}{u} du = o\left(\frac{A_{n,\left(\frac{1}{t}\right)}}{\gamma\left(\frac{1}{t}\right)}\right) \text{ as } t \to +0, \qquad (7)$$

uniformly in set E defined in the interval (-1, 1), $0 < \eta < 1$, then the Legendre series (1) is summable by triangular matrix method (T) uniformly in E to the sum f (x) which is bounded in E, where $\gamma(t)$ is positive monotonic increasing function of t provided $\gamma(n) \to \infty$ as $n \to \infty$ and $T = (a_{n,k})$ be an infinite lower triangular matrix such that the element ($a_{n,k}$) be positive, monotonic non decreasing with $k \le n$, $A_{n,u} = \int_0^u (a_{n,n-u})\, du$, for $0 \le u \le n$ and $A_{n,n} = 1\ \forall n$.

Lemmas:

Following lemmas are required for the proof of the theorem.

Lemma 1: The condition (7) implies that

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left(\frac{t}{\gamma(\frac{1}{t})}\right) \text{ as } t \to +0. \tag{8}$$

Prof. We have

$$\sigma(t) = \int_t^\eta \frac{|\psi(u)| A_{n,(\frac{1}{u})}}{u}\, du = o\left(\frac{A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right)$$

therefore

$$\int_0^t |\psi(u)| A_{n,(\frac{1}{u})}\, du = \int_0^t \frac{|\psi(u)| A_{n,(\frac{1}{u})}}{u}\, u\, du$$

$$= [u\,\sigma(u)]_0^t + \int_0^t \sigma(u)\, du$$

$$= o\left(\frac{t A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right) + o\left(\frac{A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right) \int_0^t du$$

$$= o\left(\frac{t\, A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right).$$

Since

$$\int_0^t |\psi(u)| A_{n,(\frac{1}{u})}\, du \ge A_{n,(\frac{1}{t})} \int_0^t |\psi(u)|\, du.$$

We get

$$\int_0^t |\psi(u)|\, du = o\left(\frac{t}{\gamma(\frac{1}{t})}\right).$$

Lemma 2: Let $N_n(t) = \sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}$, than

$$|N_n(t)| = O(n) \text{ uniformly in } 0 < t < \tfrac{1}{n}. \tag{9}$$

Proof:

$$\begin{aligned}|N_n(t)| &= \left|\sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}\right| \\ &\le \sum_{k=0}^{n} a_{n,k} \frac{(k+1)|\sin t|}{|\sin\frac{t}{2}|} \\ &\le (n+1) \sum_{k=0}^{n} a_{n,k} \\ &= (n+1) A_{n,n} \\ &= O(n).\end{aligned}$$

Lemma 3: If ($a_{n,k}$) be non negative and non decreasing with $k \le n$, then for $0 \le a < b \le \infty$, $0 \le t \le \pi$ and for any n,

$$\left|\sum_{k=a}^{b} a_{n,k}\, e^{ikt}\right| = O(A_{n,\tau})$$

where τ = Integral part of $\frac{1}{t} = [\frac{1}{t}]$.

Lemma 3 may be proved by the following technique of Lemma 4.1 in Lal [1].

Lemma 4: Let $N_n(t)$ be given as in Lemma 2 and using lemma 3, we have

$$|N_n(t)| = O\left(\frac{A_{n,\tau}}{t}\right) \text{ uniformly in } \tfrac{1}{n} < t < \eta. \tag{11}$$

Proof. Since, $\sin\frac{t}{2} > \frac{t}{\pi}, 0 < t < \eta < \pi$ therefore, $\tau \le n$, we have

$$\begin{aligned}|N_n(t)| &= \left|\sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}\right| \\ &= O\left(\frac{1}{t}\right)\left| \operatorname{Im}.\sum_{k=0}^{n} a_{n,k}\, e^{i(k+1)t}\right| \\ &= O\left(\frac{1}{t}\right)\left|\sum_{k=0}^{n} a_{n,k}\, e^{ikt}\right| \left|e^{it}\right| \\ &= O\left(\frac{A_{n,\tau}}{t}\right).\end{aligned}$$

3. Proof of the Theorem

Following Prasad & Tripathi [2], the k th partial sum of Legendre series (1) is given by

$$S_k(x)-f(x)=\frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}\frac{f\{\cos(\theta-t)\}-f(\cos\theta)}{\sin\frac{1}{2}}\sin(k+1)t\sqrt{\sin(\theta-t)}\,dt+o(1)$$

uniformly in E.

where

$$0\le\eta\le\delta<1,\ x=\cos\theta,\ y=\sin\theta,\ 0<\theta<\pi,\ \theta-\pi=t,\ 0<\phi<\pi \text{ etc.}$$

Now,

$$\sum_{k=0}^{n}a_{n,k}\left(S_k(x)-f(x)\right)$$

$$=\frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}f\{\cos(\theta-t)\}-f(\cos\theta)\sin(k+1)t\sqrt{\sin(\theta-t)}\sum_{k=0}^{n}a_{n,k}\frac{\sin(k+1)t}{\sin\frac{1}{2}}\,dt+o(1)$$

$$=O\left[\int_0^{\eta}|\psi(t)|\,|N_n(t)|\,dt\right]+o(1)\ \text{uniformly in set E.}$$

$$=O\left[\int_0^{\frac{1}{n}}|\psi(t)|\,|N_n(t)|\,dt\right]+O\left[\int_{\frac{1}{n}}^{\eta}|\psi(t)|\,|N_n(t)|\,dt\right]+o(1)\ \text{uniformly in set E.}$$

$$=I_1+I_2+o(1),\ \text{say.} \tag{12}$$

In order to prove the theorem, we have to show that under our assumption

$I_1=o(1)$ and $I_2=o(1)$ as $n\to\infty$, uniformly in set E.

Now considering I_1, we have

$$I_1=O\left[\int_0^{\frac{1}{n}}|\psi(t)|\,|N_n(t)|\,dt\right]$$

$$=O(n)\int_0^{\frac{1}{n}}|\psi(t)|\ dt$$

$$=O(n)\left(\Psi\left(\tfrac{1}{n}\right)\right)$$

$$=o\left(\frac{1}{\gamma(n)}\right)$$

$$=o(1)\ \text{as } n\to\infty,\ \text{uniformly in set E.} \tag{13}$$

Again, for I_2, we have

$$I_2 = O\left[\int_{\frac{1}{n}}^{\eta} |\psi(t)|\,|N_n(t)|\,dt\right]$$

$$= O\left[\int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|\,A_{n,\tau}}{t}\,dt\right]$$

$$= o\left(\frac{A_{n,n}}{\gamma(n)}\right)$$

$$= o\left(\frac{1}{\gamma(n)}\right)$$

$= o\,(1)$ as $n \to \infty$, uniformly in set E. (14)

Collecting (12) – (14), we get

$I = o\,(1)$ as $n \to \infty$, uniformly in set E.

This completes the proof of the theorem.

4. Applications

I. Particular cases are

1. If $a_{n,k} = \frac{p_{n-k}}{P_n}$ and $\gamma(t) = \alpha(P_{[t]})\ \forall t$, result of Prasad [3] becomes the particular case of main theorem.
2. If $a_{n,k} = \frac{q_{n-k}}{Q_n}$ and $\gamma(t) = \frac{\beta(Q_{[t]})}{t\,\lambda(t)\,q_{[t]}}\ \forall t$, result of Prasad & Tripathi [2] becomes the particular case of main theorem.
3. If $a_{n,k} = \frac{p_{n-k}}{P_n}$ and $\gamma(t) = \frac{P_{[t]}}{t\,p_{[t]}}\ \forall t$, result of Tripathi [5] becomes the particular case of main theorem.

II. Following corollary can be derived easily from our theorem.

Let a sequence $\{p_n\}$ be defined as $p(u)$, monotonic decreasing and strictly positive for $u \geq 0$;

$$P(u)=\int_0^u p(x)\,dx\,;\; p_n = p(n)\,.$$

If

$$\int_{\frac{1}{n}}^{\eta} \frac{|\psi(u)|P(\frac{1}{u})}{u}\,du = o\left(\frac{P_n}{\gamma(n)}\right),\; 0<\eta<1$$

uniformly in set E defined in the interval (-1, 1), then the Legendre series (1) is summable by Nörlund method uniformly in E to the sum f (x), where $\gamma(t)$ is positive monotonic increasing function of t provided $\gamma(n)\to\infty$ as $n\to\infty$.

REFERENCES

1. ***Lal, Shyam***: On the degree of approximation of conjugate of function belong to Weighted to $W(L^p,\xi(t))$ class by matrix summability means of conjugate series of Fourier series, Tamkang J. Math., 31(4), 279-288, 2000.
2. ***Prasad, K. and Tripathi, L. M.***: On the uniform Nörlund summability of Legendre series, Indian J. Pure Appl. Math., 6(6), 687 - 694, 1975.
3. ***Prasad, Rajendra:*** On the uniform Nörlund summability of Legendre series, Indian J. Pure Appl. Math., 10(10), 1298 - 1302, 1979.
4. ***Töeplitz, O.***: über die lineare Mittelbildungen, prace mat. - fiz., 22, 113-118, 1911.
5. ***Tripathi, L. M.***: On the Nörlund summability of Legendre series, Progress of Mathematics, 11, 85-91, 1977.

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