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The Schrödinger equation associated to 2nd order linear differential equation

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Abstract: We determine the Schrödinger equations associated to Hermite and Laguerre differential equations, hoping that the process here exhibited may be useful in quantum mechanics.

1. Introduction:

It is known [1,2] that the 2nd order linear differential equation :

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \tag{1}$$

can be written as an Schrödinger-like equation:

$$\frac{d^2W}{dx^2}+J(x)W=0 \tag{2}$$

via the following change of variable:

$$y=W\exp\left(-\frac{1}{2}\int^{x}P(t)dt\right) \tag{3}$$

such that:

$$J(x)=Q(x)-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4} \tag{4}$$

We here shall apply this procedure to Laguerre and Hermite equations, which has didactic value in the teaching of the elementary quantum mechanics

2. The Schrodinger type equations associated to Hermite and Laguerre equations.

The Hermite equation is given by [1, 2]:

$$y''-2xy'+2ny=0 \quad , \quad n=0,1,2,\ \dots \tag{5}$$

and its corresponding polynomial solution is denoted by $H_n(x)$. By comparison of (1) with (5) we see that $P = -2x$ and $Q = 2n$, then $J = 2n + 1 - x^2$ and thus the Schrödinger equation (2) adopts the form

$$-\frac{1}{2}W'' + \frac{x^2}{2}W = \left(n + \frac{1}{2}\right)W \tag{6}$$

for the potential $\frac{x^2}{2}$ of the harmonic oscillator in natural units ($\hbar = m = \omega = 1$), resulting thus the energy spectrum $\left(n + \frac{1}{2}\right)$ for the stationary status. The equation (3) implies $W \propto H_n \exp\left(-\frac{x^2}{2}\right)$, then the normalization of the waves functions leads to final result [3–5] :

$$\psi_n(x) = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}} H_n(x)\exp\left(-\frac{x^2}{2}\right). \tag{7}$$

The associated Laguerre equation has the structure [1, 2]:

$$y'' + \frac{k+1-x}{x}\, y' + \frac{N}{x}\, y = 0 \tag{8}$$

and the polynomials $L_N^K(x)$ represent their respective solutions. From (1) and (8) it is clear that $P = \frac{k+1-x}{x}$ with $Q = \frac{N}{x}$, then (2), (3) and (4) give us the expressions :

$$W'' + \left(-\frac{1-K^2}{4x^2} + \frac{K+1+2N}{2x} - \frac{1}{4}\right)W = 0 \tag{9}$$

with

$$W \propto x^{\frac{K+1}{2}} e^{-\frac{x}{2}} \; L_N^K(x). \tag{10}$$

In (9) is not evident the corresponding potential, thus we make the changes :

$$K = 2l + 1 \;,\; N = n - l - 1 \;,\; x = \frac{2r}{bn} \;,\; b = \frac{4\pi\varepsilon_0}{Ze^2} \tag{11}$$

then (9) takes the known form for the Coulomb potential ($\hbar = m = 1$) :

$$-\frac{1}{2}\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right]W - \frac{Ze^2}{4\pi\varepsilon_0 r}\; W = \frac{Z^2}{32\pi^2\varepsilon_0^2}\,\frac{1}{n^2}W \tag{12}$$

where n and l denote the principal and orbital quantum numbers, respectively.Therefore, (10) and (11) imply the normalized radial wave functions [3,4, 6]:

$$\psi_{nl}(r) = \left(\frac{2r}{n}\right)^{l+1}\left[\frac{(n-l-1)!}{(n+l)!}\right]^{1/2} \frac{e^{-\frac{r}{bn}}}{b^{l+3/2}}\; L_{n-l-1}^{2l+1}\left(\frac{2r}{bn}\right). \tag{13}$$

If in (9) we use changes of variables different to (11), then it is easy to show that (9) reproduces the radial part of the Schrödinger equation for the Morse and two-dimensional harmonic oscillator potentials [7].

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