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The $(E,1)(C,1)$ summability of the conjugate series of a Fourier series

BINOD PRASAD DHAKAL

Department of Mathematics
Butwal Multiple Campus,
Trubhuvan University, NEPAL.

SHYAM LAL

Department of Mathematics
Banaras Hindu University,
Varanasi–221 005 INDIA.
E-mail : shyam_lal@rediffmail.com

Abstract: In this paper, a new theorem on $(E, 1)$, $(C, 1)$ summability of the conjugate series of a Fourier series has been proved.

Key words and phrases: $(E,1)$ $(C,1)$ Summability means, Fourier series, Conjugate series of a Fourier series, periodic function, n[th] partial sum.

1. Definitions and Notations

Let $f(t)$ be 2π periodic function and integrable over $(-\pi,\pi)$ in the sense of Lebesgue. Then its "Fourier series" is given by

$$f(t)=\frac{1}{2}a_o+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(t) \tag{1.1}$$

The series

$$\sum_{n=1}^{\infty}(a_n\sin nt-b_n\cos nt)=-\sum_{n=1}^{\infty}B_n(t) \tag{1.2}$$

is called the "conjugate series" of the Fourier series (1.1),
where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt$$

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and

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \end{aligned}\right\} n = 1, 2, 3, \dots$$

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose n^{th} partial sum is given by $S_n = \sum_{i=0}^{\infty} u_i$.

We write

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k = \text{Cesàro means (C, 1) of sequence } \{S_n\}.$$

If

$$\sigma_n \to S, \text{ as } n \to \infty.$$

where S is a finite number, then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesàro means method $(C, 1)$ to S. It is denoted by

$$\sigma_n \to S\,(C,1), \text{ as } n \to \infty \text{ (Hardy, 1913)}.$$

Next,

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} S_k = \text{Euler means } (E, 1) \text{ of sequences } \{S_n\}.$$

If

$$E_n^1 \to S \quad \text{as } n \to \infty.$$

then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_i$ is said to be summable by Euler means method $(E, 1)$ to S. It is denoted by

$$E_n^1 \to S\,(E, 1) \quad \text{as } n \to \infty \text{ (Hardy, 1949, p. 180)}.$$

The E_n^1 trsnsformation of $\{\sigma_n\}$, denoted by $t_n^{E_1,C_1}$, is defined by

$$\begin{aligned} E_n^{E_1,C_1} &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k. \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k} S_r. \end{aligned}$$

If $t_n^{E_1,C_1} \to S$, as $n \to \infty$.

then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E, 1\ (C, 1)$ means method to S. It is denoted by

$$t_n^{E_1, C_1} \to S\ (E, 1)\,(C, 1) \text{ as } n \to \infty.$$

Thus, if $(E, 1)$ transform is superimposed on $(C, 1)$ transform of sequence $\{S_n\}$, a new transformation $(E, 1)\,(C, 1)$ is obtained.

Here

$$S_n \to S \ \Rightarrow \sigma_n(S_n) \to S, \text{ as } n \to \infty \text{ since } (C, 1) \text{ method is regular.}$$

$$\Rightarrow E_n^1(\sigma_n) \to S, \text{ as } n \to \infty \text{ since } (E, 1) \text{ method is regular.}$$

$$\Rightarrow (E,1)\,(C, 1) \text{ method is regular.}$$

We use following notations

$$\psi(t) = f(x+t) - f(x-t) \tag{1.3}$$

$$\Psi(t) = \int_0^t |\psi(u)|\,du \tag{1.4}$$

$$N_n^{E_1, C_1}(t) = \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}. \tag{1.5}$$

2. Introduction

Quite a good amount of works are known on Summability of a Fourier series and its allied series. Versaney (1959) has dicussed $(H,1)\,C_1$ summability on sequence of the Fourier coefficient. Naturally, we have to consider other product summability method of the form $(E,1)$ $(C, 1)$.

Recently, Dhakal & Lal (2007) have proved a theorem on $(E,1)\,(C,1)$ summability of a Fourier series in the following form.

Theorem A: If

$$\Phi(t) = \int_0^t |\phi(u)|\,du = \mathrm{o}\left(\frac{t\xi\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\right), \quad \text{as } t \to +\mathrm{o}. \tag{2.1}$$

provided $\xi(t)$ is a prositive monotonic decreasing function of t such that $\dfrac{t\xi\left(\frac{1}{t}\right)}{\log\frac{1}{t}}$ increases monotonically as $t \to +\mathrm{o}$, then the Fourier series (1,1) is summable by $(E,1)\,(C,1)$ method to $f(x)$ at $t = x$.

3. Theorem: *The purpose of this paper is to study the conjugate series of the Fourier series by $(E,1)\,(C,1)$ summability method. In fact, we prove following theorem:*

Theorem: If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left(\frac{t\xi\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\right), \text{ as } t \to +o. \tag{3.1}$$

then the conjugate series of the Fourier series (1.2) is summable by $(E,1)\,(C,1)$ method at $t = s$, to

$$\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\tfrac{t}{2}\, dt,$$

provided this integral exist in sense of Lebesgue.

4. Proof of the theorem

Following Lal (1997) and using Roemann-Lebesgue theorem, nth partial sum $S(x)$ of conjugate series (1.2) of Fourier series at $t = x$ is given by

$$\tilde{S}_n(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t - \cos\frac{t}{2}}{\sin\frac{t}{2}}\, dt$$

$$\tilde{S}_n(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt - \frac{1}{2\pi}\int_0^\pi \psi(t)\cot\tfrac{t}{2}\, dt$$

$$\tilde{S}_n(x) - \left(-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\tfrac{t}{2}\, dt\right) = \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\cos\left(n+\frac{t}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

$(C,1)$ transform of $\tilde{S}(x)$ i.e. $\sigma(x)$

$$\sigma_n(x) - \left(-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\tfrac{t}{2}\, dt\right) = \frac{1}{2(n+1)\pi}\int_0^\pi \psi(t)\sum_{k=0}^{n}\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

$(E,1)$ transform of $\tilde{\sigma}_n(x)$ i.e., $\tilde{t}_n^{E_1,C_1}$

$$\tilde{t}_n^{E_1,C_1} - \left(-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\tfrac{t}{2}\, dt\right) = \int_0^\pi \psi(t)\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

$$= \int_0^\pi \psi(t)\,\tilde{N}_n^{E_1,C_1}(t)\, dt$$

$$= \left(\int_0^{\frac{1}{n}} \psi(t)\,\tilde{N}_n^{E_1,C_1}(t)\, dt + \int_{\frac{1}{n}}^{\delta} \psi(t)\,\tilde{N}_n^{E_1,C_1}(t)\, dt + \int_{\delta}^{\pi} \psi(t)\,\tilde{N}_n^{E_1,C_1}(t)\, dt\right)$$

$$= (I_1 + I_2 + I_3). \tag{4.1}$$

Since conjugate function exist, therefore

$$\frac{1}{2\pi}\int_0^{\frac{1}{n}}\psi(t)\cot\tfrac{t}{2}\,dt = o(1) \quad \text{as } n\to\infty.$$

(4.2)
$$\frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\left(\frac{1}{2\pi}\int_0^{\frac{1}{n}}\psi(t)\cot\tfrac{t}{2}\,dt\right) = o(1) \quad \text{as } n\to\infty.$$

We have, for $0<t<\dfrac{1}{n}$

$$\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\cos\left(r+\frac{1}{2}\right)t-\cos\frac{t}{2}}{\sin\frac{t}{2}}$$

$$= -\frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\sin(r+1)\frac{t}{2}\sin\frac{rt}{2}}{\sin\frac{t}{2}}$$

$$= -\frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\sum_{m=1}^{r}\sin mt$$

$$\leq \frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\sum_{m=0}^{r}1$$

$$= \frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}r$$

$$= \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}k$$

$$= \frac{n}{4\pi}$$

(4.3)
$$= O(n)$$

Using (1.5), (3.1), (4.2) and (4.3) we have

$$|I_1| = \int_0^{\frac{1}{n}}|\psi(t)|\,|\tilde{N}^{E_1C_1}(t)|\,dt$$

$$= \int_0^{\frac{1}{n}}|\psi(t)|\left|\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|dt$$

$$= \int_0^{\frac{1}{n}} |\psi(t)| \left| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t - \cos\frac{t}{2}}{\sin\frac{t}{2}} \right| dt$$

$$+ \int_0^{\frac{1}{n}} |\psi(t)| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \cot\tfrac{t}{2}\, dt$$

$$= \int_0^{\frac{1}{n}} |\psi(t)|\, O(n)\, dt + \frac{1}{2^n\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \left(\frac{1}{2\pi} \int_0^{\frac{1}{n}} \psi(t) \cot\tfrac{t}{2}\, dt \right)$$

$$= O(n)\, \Psi\left(\frac{1}{n}\right) + o(1)$$

$$= O(n)\, o\left(\frac{\xi(n)}{n \log n}\right) + o(1)$$

$$= o\left(\frac{\xi(n)}{\log n}\right) + o(1)$$

(4.4)1.1. $= 0(1)$, by the hyperthesis of the Theorem

Also, for $\frac{1}{n} < t < \delta$

$$\frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

$$= \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{\sin(r+1)t}{(k+1)\sin^2\frac{t}{2}}$$

$$\leq \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{|\sin(r+1)t|}{(k+1)\left|\sin^2\frac{t}{2}\right|}$$

$$= \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)\left|\sin^2\frac{t}{2}\right|}$$

$$= \frac{\pi}{2^{n+2}t^2} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)}$$

$$= \frac{\pi}{2^{n+2}t^2} \left(\frac{2^{n+1}-1}{n+1} \right)$$

$$= \frac{\pi}{2(n+1)t^2}\left(1-\frac{1}{2^{n+1}}\right)$$

$$\leq \frac{\pi}{nt^2}$$

$$= O\left(\frac{1}{nt^2}\right) \tag{4.5}$$

Using (3.1) and (4.5), we have

$$|I_2| = \int_{\frac{1}{n}}^{\delta} |\psi(t)| \left| \tilde{N}_n^{E_1C_1}(t) \right| dt$$

$$= O\left(\frac{1}{n}\right) \int_{\frac{1}{n}}^{\delta} \frac{|\psi(t)|}{t^2} dt$$

$$= O\left(\frac{1}{n}\right) \left[\left(\frac{\Psi(t)}{t^2} \right)_{\frac{1}{n}}^{\delta} - 2\int_{\frac{1}{n}}^{\delta} \frac{\Psi(t)}{t^3} dt \right]$$

$$= O\left(\frac{1}{n}\right) \left[o\left(\left(\frac{\xi\left(\frac{1}{t}\right)}{t \log \frac{1}{t}} \right)_{\frac{1}{n}}^{\delta} \right) + o\left(\int_{\frac{1}{n}}^{\delta} \frac{t\,\xi\left(\frac{1}{t}\right)}{t^3 \log \frac{1}{t}} dt \right) \right]$$

$$\leq O\left(\frac{1}{n}\right) \left[o\left(\frac{\xi\left(\frac{1}{\delta}\right)}{\delta \log \frac{1}{\delta}} \right) + o\left(\frac{n\xi(n)}{\log n} \right) + o\left(\frac{\xi(n)}{n \log n} \right) \int_{\frac{1}{n}}^{\delta} \frac{1}{t^3} dt \right]$$

by Second Mean value theroem for integral calculas.

$$= o\left(\frac{\xi\left(\frac{1}{\delta}\right)}{n\delta \log \frac{1}{\delta}} \right) + o\left(\frac{\xi(n)}{\log n} \right) + o\left(\frac{\xi(n)}{\delta^2 n^2 \log n} \right)$$

$$= o(1) + o(1) + o(1) \quad \text{as } n \to \infty.$$

$$= o(1), \quad \text{as } n \to \infty. \tag{4.6}$$

Lastly, By the Riemann-Lebesgue theorem and the regularity conditions of (E,1), (C,1) summability, we have

$$|I_3| = \int_{\delta}^{\pi} |\psi(t)| \left| \tilde{N}_n^{E_1C_1}(t) \right| dt$$

$$= \int_{\delta}^{\pi} |\psi(t)| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt$$

$$= \frac{1}{2^{n+2}\pi} \int_{\delta}^{\pi} \psi(t) \sum_{k=0}^{n} \binom{n}{k} \frac{\sin(r+1)t}{(k+1)\sin^2 \frac{t}{2}} dt$$

$$= o(1) \text{ as } n \to \infty. \tag{4.7}$$

Collecting (4.1), (4.4), (4.6) and (4.7), we have

$$\tilde{t}_n^{E_1 . C_1} - \frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \tfrac{t}{2}\, dt = o(1) \quad \text{as} \quad n \to \infty \tag{4.8}$$

This completes the proof of the theorem

Remark: It is remarkable that our theorem is analogous to theorem Dhakal & Lal (2007) for a Fourier series.

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