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Two Different Ways to Show a Function is an A_1 Weight Function

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Abstract

In this paper, we briefly discuss the theory of weights and then define A_1 and A_p weight functions. Finally we explore two different ways to show a function is an A_1 weight function.

Introduction

The theory of weights play an important role in various fields such as extrapolation theory, vector-valued inequalities and estimates for certain class of non linear differential equation. Moreover, they are very useful in the study of boundary value problems for Laplace's equation in Lipschitz domains. In 1970, Muckenhoupt characterized positive functions w for which the Hardy-Littlewood maximal operator M maps $L^p(R^n, w(x)dx)$ to itself. Muckenhoupt's characterization actually gave the better understanding of theory of weighted inequalities which then led to the introduction of A_p class and consequently the development of weighted inequalities. Before we explore two different ways to show a function is and A_1 weight, some definitions are in order.

Definition: A locally integrable function on R^n that takes values in the interval $(0,\infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero.

We use the notation $w(E) = \int_E w(x)dx$ to denote the w-measure of the set E and we reserve the notation $L^p(R^n,w)$ or $L^p(w)$ for the weighted L^p spaces. We note that $w(E) < \infty$ for all sets E contained in some ball since the weights are locally integrable functions.

Definition: The uncentered Hardy-Littlewood maximal operators on R^n over balls B is defined as

$$M(f)(x) = \sup_{x\in B} \operatorname{Avg}_{B}|f| = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|dy.$$

Similarly the uncentered Hardy-Littlewood maximal operators on R^n over cubes Q is defined as

$$M_c(f)(x) = \sup_{x\in Q} \operatorname{Avg}_{Q}|f| = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy.$$

In each of the definition above, the suprema are taken over all balls B and cubes Q containing the point x. H-L maximal functions are widely used in Harmonic Analysis. For the details about the H-L maximal operators, see [2].

Definition: A positive measure dμ is called doubling measure if for some positive constant $C < \infty$,

$$\mu(2B) \le C\mu(B)$$

for all balls B. This means that the size of a ball with certain radius can be controlled by the ball of half of the given radius.

Definition: A function $w(x) \ge 0$ is called an A_1 weight if there is a constant $C_1 > 0$ such that

$$M(w)(x) \le C_1\, w(x)$$

where M(w) is uncentered Hardy-Littlewood Maximal function given by

$$M(w)(x) = \sup_{x\in B} \frac{1}{|B|}\int_B w(t)dt.$$

If w is an A_1 weight, then the quantity (which is finite) given by

$$[w]_{A_1} = \sup_{Q\ cubes\ in R^n} \left(\frac{1}{|Q|}\int_Q |w(t)|dt\right) ||w^{-1}||_{L^\infty(Q)}$$

is called the A_1 characteristic constant of w.

Definition: Let $1 < p < \infty$. A weight w is said to be of class A_p if $[w]_{A_p}$ is finite where $[w]_{A_p}$ is defined as

$$[w]_{A_p} = \sup_{Q\ cubes\ in\ R^n} \left(\frac{1}{|Q|}\int_Q |w(x)|dx\right)\left(\frac{1}{|Q|}\int_Q |w(x)|^{\frac{-1}{p-1}}dx\right)^{p-1}.$$

We remark that in the above definition of A_1 and A_p one can also use set of all balls in R^n instead of all cubes in R^n. Readers are suggested to read [1] for motivation, properties of A_p weights and much more about the A_p weights.
Consider the following function:

$$u(x) = \begin{cases} log\,\frac{1}{|x|}, & |x| < \frac{1}{e} \\ 1, & otherwise. \end{cases}$$

We introduce two different ways to show that the above function is an A_1 weight function.

First approach: Set $v_n = \frac{|s(0,1)|}{n}$, where $|s(0,1)|$ is the surface area of the unit ball. Let $B_\epsilon = \{x \in R^n : |x| \le \epsilon\}$. Then

$$\int_{B(0,1/e)} log\left(\frac{1}{|x|}\right) dx = \lim_{\epsilon\to 0} \int_{B(0,1/\epsilon)\setminus B_\epsilon} \log\left(\frac{1}{|x|}\right)\, dx$$

$$= \lim_{\epsilon\to 0} \int_\epsilon^{1/e} \log\left(\frac{1}{r}\right) r^{n-1}\, dr |B(0,1/e)\setminus B_\epsilon|$$

$$= \frac{1}{ne^n}\,(\log(e)+1)|B(0,1/e)|$$

$$= \frac{1}{ne^{2n}}\,(\log(e)+1)v_n$$

$$\le 2v_n.$$

Note that $\sup_{B_{1/e}}\left(\log\left(\frac{1}{|x|}\right)\right)^{-1} = ||u^{-1}||_{L^\infty(B_{1/e})} \le 1$. Let $T_1 := \{B(x_0,R) : |x_0| \ge 3R\}$ and $T_2 := \{B(x_0,R) : |x_0| < 3R\}$. *For all the balls in* T_1 *we have* $2R \le |x_0| - R < |x| <$ $|x_0| + R$. In the case $\frac{1}{e} \le 2R$, we have u=1 and so $[u]_{A_1} \le 1$. Note that

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} = \left(\frac{1}{|B|}\int_{B\cap B_{1/e}} u(x)dx\right) ||u^{-1}||_{L^\infty(B\cap B_{1/e})}$$

$$+ \left(\frac{1}{|B|}\int_{B\setminus B_{1/e}} u(x)dx\right) ||u^{-1}||_{L^\infty(B\setminus B_{1/e})}.$$

For $2R < \frac{1}{e} < |x_0| + R$, we have $B\cap B_{1/e} \ne \phi$ and $B\setminus B_{1/e} \ne \phi$. In this case we have,

$$\left(\frac{1}{|B|}\int_{B\cap B_{1/e}} u(x)dx\right) ||u^{-1}||_{L^\infty(B\cap B_{1/e})} \le \left(\frac{1}{|B|}\int_{B_{1/e}} u(x)dx\right) ||u^{-1}||_{L^\infty(B_{1/e})} \le 2v_n$$

and

$$\left(\frac{1}{|B|}\int_{B\setminus B_{1/e}} u(x)dx\right) ||u^{-1}||_{L^\infty(B\setminus B_{1/e})} \le 1.$$

Thus we have, $[u]_{A_1} \leq 1 + 2\nu_n$.

In the case, $|x_0| + R < \frac{1}{e}$ one has $B \cap B_{1/e} \neq \phi$ and $B \backslash B_{1/e} = \phi$ and so

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} = \left(\frac{1}{|B|}\int_{B_{1/e}} u(x)dx\right) ||u^{-1}||_{L^\infty(B_{1/e})} \leq 2\nu_n.$$

Putting all together, we get

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

for all balls in T_1.

For the balls in T_2 we have, $0 \leq |x| \leq |x_0| + R < 4R$. Using the same calculation as above, one gets for $\frac{1}{e} < |x_0| + R$,

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

and

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

for $|x_0| + R < \frac{1}{e}$. In the case, $\frac{1}{e} \leq |x_0| - R$ we get $[u]_{A_1} \leq 1$. Thus,

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

for all balls in T_2. Thus, $u \in A_1$.

Second approach: We need to show that there exists M>0 such that

$$\frac{1}{|B|}\int_B u(x)dx \leq M \ \text{ess.}\inf_{x\in B} u(x), \qquad \forall B \subset R^n \qquad (1).$$

First, we assume that $B(x_0, R)$ is such that $|x_0| > 3R$. Let's say this is of type I.

Case 1: $|x_0| \leq \frac{1}{16}$. Then $\leq \frac{1}{16}$. We have,

$$\frac{2}{3}|x_0| \leq |x_0| - R \leq |x| \leq |x_0| + R \leq \frac{4}{3}|x_0| \leq \frac{1}{e}, \qquad \forall x \in B.$$

Thus,

$$1 \leq \ln\frac{1}{|x_0|+R} \leq u(x) \leq \ln\frac{1}{|x_0|-R}, \ \forall x \in B.$$

Hence,

$$\frac{1}{|B|}\int_B u(x)dx \leq \ln\frac{1}{|x_0|-R} \leq \ln\frac{3}{2|x_0|}.$$

Moreover we have,

$$\ln\frac{3}{2|x_0|} = \ln\frac{3}{4|x_0|} + \ln 2 \leq 2\ln\frac{3}{4|x_0|} \leq 2\ln\frac{1}{|x_0|+R} \leq 2u(x), \quad \forall x \in B.$$

Therefore, $\frac{1}{|B|}\int_B u(x)dx \leq M$ ess. $\inf_{x\in B} 2u(x)$.

Case 2: $|x_0| > \frac{3}{16}$. Then $|x| \geq |x_0| - R \geq \frac{2}{3}|x_0|$. Thus,

$$1 \leq u(x) \leq \max\left(1, \ln\frac{16}{3}\right) = \ln\frac{16}{3}.$$

$$\frac{1}{|B|}\int_B u(x)dx \leq \ln\frac{16}{3} \leq M \text{ ess.} \inf_{x\in B} u(x), M = \ln\frac{16}{3}.$$

Therefore (1) holds when B is of type I.

Secondly consider the case B is such that $B(x_0, R)$ is such that $|x_0| < 3R$. Let's call this is of type II. In this case one has, $B(x_0, R) \subset B(0, 5R)$. Note that,

$$0 < a := \int_{B^n}[u(x) - 1]dx = \int_{B(0,1/e)}[u(x) - 1]dx < \infty.$$

Case 1: $5R > \frac{1}{5}$. We have

$$\frac{1}{|B|}\int_B u(x)dx = 1 + \frac{1}{|B|}\int_B [u(x) - 1]dx$$
$$\leq 1 + \frac{1}{v_n R^n} a \leq 1 + \frac{(5e)^n}{v_n} a.$$

Thus, (1) is satisfied with $M = 1 + \frac{(5e)^n}{v_n} a$.

Case 2: $5R \leq \frac{1}{5}$. We have

$$\frac{1}{|B|}\int_B u(x)dx \leq \frac{1}{v_n R^n}\int_{B(0,5R)} u(x)dx =: J.$$

Let $x = 5Ry$. We have,

$$J = \frac{1}{v_n R^n}\int_{B(0,1)} \ln\frac{1}{5R|y|}(5R)^n dy$$
$$= \frac{5^n}{v_n}\int_{B(0,1)}\left(\ln\frac{1}{5R} + \ln\frac{1}{|y|}\right)dy$$
$$= 5^n\left(\ln\frac{1}{5R} + b\right)$$

where $b := \int_{B(0,1)} \ln\frac{1}{|y|}\, dy$. Since $\ln\frac{1}{5R} \nearrow \infty$ as $R \to 0$, there exists c>0 such that

$$b \leq c \ln \frac{1}{5R}, \quad \forall R < \frac{1}{5e}.$$

Here we have used the fact that $1 \leq \ln \frac{1}{5R}, \ \forall R < \frac{1}{5e}$. Therefore,

$$\frac{1}{|B|} \int_B u(x)dx \leq J \leq 5^n \, (1 + \mathrm{c}) \ln \frac{1}{5R} \leq 5^n \, (1 + \mathrm{c}) \ln \frac{1}{|x|}, \qquad \forall x \in B(x_0, R).$$

Thus (1) holds for $M = 5^n \, (1 + \mathrm{c})$. So in all possible cases, we have shown that the given function satisfies the criteria to be an A_1 weight function.

REFERENCES

[1] Loukas Grafakos, Modern Fourier Analysis, Second Edition, Springer 2009.

[2] R. Ba~nelos and C. N. Moore, Probabilistic Behavior of Harmonic Functions, Birkhauser Verlag,1991.

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