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Uniform Version of the Wiener-Tauberian Theorem for Real Line

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Abstract: The wiener-Tauberian theorem for $\mathbb{R}$ says that the closed translation invariant subspace generated by an $f \in L^1(\mathbb{R})$ is $L^1(\mathbb{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we prove a uniform version of this result for $\mathbb{R}$.

Key words : Wiener-Tauberian theorem, locally compact abelian groups, translation invariant subspace.

1. Introduction :

The general Tauberian theorem proved by N.Wiener [11] says that if $g \in L^1(\mathbb{R})$ is a uniqueness function in the sense that its Fourier transform $\hat{g}$ vanishes nowhere on $\mathbb{R}$ (and thus the closed translation invariant subspace generated by g is $L^1(\mathbb{R})$ and $\phi \in L^\infty(\mathbb{R})$ is such that $g * \phi(x) \to A\hat{g}(0)$ (A is a complex number) as $x \to \infty$ then for every $f \in L^1(\mathbb{R}), f * \phi(x) \to A\hat{f}(0)$ as $x \to \infty$. Here the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by $\hat{f}(y) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi\, ixy)\, dx$ for $y \in \mathbb{R}$. For a generalization of the above result for locally compact abelian group see [1], [4] and [7]. Various analogues of the Wiener-Tabuerian theorem for non-abelian groups are given by Ehrenpreis and Mautner [2], Kaniuth and Steiner [5], Hauenschild, Kaniuth and Kumar [3], Sitaram [9] and others, see [8] for survey.

In section 2, we obtain uniform version of the Wiener-Tauberian theorem for locally compact abelian group $\mathbb{R}$ replacing $\{g\},\{f\}$ by suitable subsets of $L^1(\mathbb{R}),\{\phi\}$ by suitable subsets of $L^\infty(\mathbb{R})$. The techniques include figuringout equicontinuous subsets of $L^\infty(\mathbb{R})$ in $\| \;\|_\infty$ − topology.

We continue these investigations and prove a uniform version of the wiener-Tauberian theorem as given in Reiter and Stegeman.

2. Uniform Version

Let $\mathbb{R}$ be a locally compact abelian group with Haar measure μ. Here in this case Haar measure is ordinary Lebesgue measure. For basic notations and terminology, we refer to [4]. For $x \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, let f_x, be the translate of f. Let $\phi_f : \mathbb{R} \to L^\infty(\mathbb{R})$ be defined by $\phi_f(x) = f_x$, $x \in \mathbb{R}$, $f \in L^\infty(\mathbb{R})$. We denote by S_1 and S_∞, the unit balls of $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ respectively.

Define $\mathsf{U} = \{g \in L^1(\mathbb{R}) : \text{for } a \in B \cap C, g * a = 0 \Rightarrow a = 0)$.

Where B and C are the sets of bounded and continuous function on $\mathbb{R}$ respectively.

The following result 2.3 may be thought of as a uniform version of the Wiener-Tauberian theorem as given in Widder ([11], theorem 3.1).

Definition 2.1. Let $\mathcal{F}$ be a collection of functions on a metric space X with metric ρ to a metric space Y with metric ρ^1. We say that $\mathcal{F}$ is uniformly equicontinuous if to every $\in > 0$ corresponds a $\delta > 0$ such that $\rho^1(f(x), f(y)) < \in$ for every $f \in \mathcal{F}$ and for all pairs of points x, y with $\rho(x,y) < \delta$.

Example 2.2 : If $X = \mathbb{R}$, $Y = L^1(\mathbb{R})$, $\mathcal{F}$ a set of functions on X to Y which is

(i) Equicontinuous at some point and
(ii) Translation-invariant i.e., for $f \in \mathcal{F}, x \in \mathbb{R}$, $f_x \in \mathcal{F}$, then $\mathcal{F}$ is uniformly equicontinuous.

For $x, y \in \mathbb{R}$ with $|y - x| < \delta$ we have

$$\|f(x) - f(y)\| = \|f_x(0) - f_x(y - x)\|$$
$$= \|f_{x-z}(z) - f_{x-z}(z + y - x)\| < \in$$

So equicontinuity of $\mathcal{F}$ at any point is equivalent to uniform equicontinuity of $\mathcal{F}$,

Theorem 2.3 : *Let $H \subset L^1(\mathbb{R})$ be such that*

(i) $\{\varphi_h : h \in H\}$ given by $\varphi_h(x) = h_x$, $x \in \mathbb{R}$ is uniformly equicontinuous.
(ii) There exists $h_0 \in S_1$ with $|h(t)| \le |h_0(t)|$ for all $h \in H$ and for all $t \in \mathbb{R}$

Let $g \in S_1 \cap \mathsf{U}$. Let $\mathcal{U} \subset S_\infty$ be a family of bounded continuous function on $\mathbb{R}$.

Suppose that $g * a(x) \to 0$ as $x \to \infty$ uniformly for a in $\mathcal{U}$ then $h * a(x) \to 0$ as $x \to \infty$ uniformly for h in H and a in $\mathcal{U}$.

Proof: Assume on the contrary. Then there must exists $\delta > 0$ such that for every n there exists $x_n \in \mathbb{R}$ with $x_n > n$, $h_n \in H$ and $a_n \in \mathcal{U}$ satisfying $|h_n * a_n(x_n)| > \delta$.

Now consider, $g * (h_n * a_n) = (g * h_n) * a_n$

Let us consider the sequence

$$s_n(x) = (h_n * a_n)(x + x_n), n = 1, 2, 3, \ldots, -\infty < x < \infty.$$

We shall show that it is bounded and equicontinuous on $\mathbb{R}$.

Now,
$$\begin{aligned} \|s_n\|_\infty &= \|(h_n * a_n)_{x_n}\|_\infty \\ &= \|h_n * a_n\|_\infty \\ &\le \|h_n\|_1 \; \|a_n\|_\infty \le \|h_0\|_1 \le 1. \end{aligned}$$

$\therefore$ s_n is bounded on $\mathbb{R}$.

Also for $x, y \in \mathbb{R}$.

$$\begin{aligned} |s_n(x) - s_n(y)| &= |(h_n * a_n)(x + x_n) - (h_n * a_n)(y + x_n)| \\ &= \left| \int_{-\infty}^{\infty} \{h_n(x + x_n - t)\, a_n(t) - h_n(y + x_n - t)\, a_n(t)\} \right| \\ &\le \int_{-\infty}^{\infty} |h_n(x + x_n - t) - h_n(y + x_n - t)| dt \end{aligned}$$

Since $x \to h_x$ is uniformly equicontinuous on $\mathbb{R}$ to $L^1(\mathbb{R})$, so given $\in > 0 \; \exists \delta' > 0$ such that for $|r_1 - r_2| < \delta'$, $\|h_{r_1} - h_{r_2}\|_1 < \in$ (by taking $r_1 = x + x_n$, $r_2 = y + x_n$). So, $|s_n(x) - s_n(y)| < \in$ for $|y - x| < \delta' \Rightarrow s_n$ is equicontinuous on $\mathbb{R}$

Thus by Ascoli's Lemma [6] we now select from the sequence $s_n(x)$ a subsequence $s_{n_k}(x)$ which tends to a limit $s(x)$ pointwise as $k \to \infty$ and continuous on $\mathbb{R}$.

For each fixed $x \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$s_{n_k}(x - t) \to s(x - t), k \to \infty, \text{ and therefore}$$
$$s_{n_k}(x - t)\, g(t) \to s(x - t)\, g(t), \; k \to \infty,$$

Now for each $t \in \mathbb{R}$, $|s_{n_k}(x - t)\, g(t)| \le |g(t)|$

Thus by Lebesgue dominated convergence theorem for each $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} s_{n_k}(x-t)\, g(t)\, dt \to \int_{-\infty}^{\infty} s(x-t)\, g(t)\, dt,\ k \to \infty$$
$$= s * g(x)$$

Now,

$$\begin{aligned}\int_{-\infty}^{\infty} s_{n_k}(x-t)\, g(t)\, dt &= \int_{-\infty}^{\infty} \left(h_{n_k} * a_{n_k}\right)\left(x_{n_k}+x-t\right) g(t)\, dt \\ &= \left(h_{n_k} * a_{n_k} * g\right)\left(x_{n_k}+x\right) \\ &= \left(\left(g * a_{n_k}\right) * h_{n_k}\right)\left(x_{n_k}+x\right) \\ &= \int_{-\infty}^{\infty} \left(g * a_{n_k}\right)\left(x_{n_k}+x-t\right) h_{n_k}(t)\, dt\end{aligned}$$

Put $$J_{k,x}(t) = \left(g * a_{n_k}\right)\left(x_{n_k}+x-t\right) h_{n_k}(t)$$

Since we know, $(g * a)(z) \to 0$ as $z \to \infty$ uniformly for a in $\mathcal{U}$, we have for a given $\in > 0\ \exists\ \Delta : |g * a(z)| < \in\ \forall z \geq \Delta$, and a in $\mathcal{U}$. Therefore

$$\left|\left(g * a_{n_k}\right)\left(x_{n_k}+x-t\right)\right| < \in \text{ for } x_{n_k}+x-t \geq \Delta.$$

Thus for fixed x and t, $g * a_{n_k}\left(x_{n_k}+x-t\right) \to 0$ as $k \to \infty$ and therefore

$$J_{k,x}(t) \to 0 \text{ as } k \to \infty.$$

Now for each k, $\left\|g * a_{n_k}\right\|_{\infty} \leq \|g\|_1 \left\|a_{n_k}\right\|_{\infty} \leq 1$ and therefore

$$\begin{aligned}|J_{k,x}(t)| &= \left|g * a_{n_k}\left(x_{n_k}+x-t\right) h_{n_k}(t)\right| \\ &\leq \left|g * a_{n_k}\left(x_{n_k}+x-t\right)\right|\left|h_{n_k}(t)\right| \\ &\leq \left|h_{n_k}(t)\right| \leq \left|h_0(t)\right|\end{aligned}$$

Thus by applying Lebesgue dominated convergence theorem for each $x \in \mathbb{R}$, $\int_{-\infty}^{\infty} J_{k,x}(t)\, dt \to 0$ as $k \to \infty$ and therefore $s * g(x) = 0$. Since $g \in \mathrm{U}$, we get $s = 0$.

But $$|s(0)| = \lim_{k\to\infty} \left|s_{n_k}(0)\right| = \lim_{k\to\infty} \left(h_{n_k} * a_{n_k}\right)\left(x_{n_k}\right) \geq \delta > 0$$

which is a contradiction. Therefore

$\int_{-\infty}^{\infty} h(x-t)\, a(t)\, dt \to 0$ as $x \to \infty$ uniformly for h in H & a in $\mathcal{U}$.

This completes the proof.

Corollary 2.4. Let $g \in S_1 \cap \mathrm{U}$ be fixed.

(i) $\mathcal{U}_1$ be a family of bounded continuous functions such that $g * a(x) \to A_a \int_{-\infty}^{\infty} g(t)dt$ as $x \to \infty$ uniformly for a in $\mathcal{U}_1$ and $M = \sup (\| a \|_\infty + |A_a|) < \infty$.

(ii) For $H_1 \subset L^1(\mathbb{R})$ suppose there exists $h_0 \in L^1(\mathbb{R})$ s.t. $|h_1(t)| \le |h_0(t)|$ for all $h_1 \in H_1$ and $t \in \mathbb{R}$ and rest is as above in the theorem, then $h * a(x) \to A_a \int_{-\infty}^{\infty} h(t)dt$ as $x \to \infty$ uniformly for h in H and a in $\mathcal{U}_1$.

Proof : Let $H \subset L^1(\mathbb{R})$ be such that

$$H = (\| h_0 \|_1 + 1)^{-1} \; H_1 = \{(\| h_0 \|_1 + 1)^{-1} \; h_1 : h_1 \in H_1\}$$

Let $h \in H$ be arbitrary then $h = (\| h_0 \|_1 + 1)^{-1} \, h_1$

Since $$|h_1(t)| \le |h_0(t)| \text{ for all } t \in \mathbb{R}$$

$$\Rightarrow (\| h_0 \|_1 + 1) \; h(t) \le |h_0(t)| \text{ for all } t \in \mathbb{R}$$

$$\Rightarrow |h(t)| \le \left| \frac{h_0(t)}{\|h_0\|_1 + 1} \right| \text{ for all } t \in \mathbb{R}$$

$$= |h_0'(t)|, \text{ where } h_0' = \frac{h_0}{\|h_0\|_1 + 1}$$

Therefore, $$\| h_0' \|_1 = \left\| \frac{h_0}{\|h_0\|_1 + 1} \right\| = \frac{\| h_0 \|}{1 + \|h_0\|_1} < 1$$

Thus $$h_0' \in S_1.$$

Taking $$\mathcal{U} = \frac{1}{M+1} \{a - A_a I : a \in \mathcal{U}_1\}$$

Now for every $f \in \mathcal{U}$,

$$f = \frac{a - A_a I}{M+1} \text{ for some } a \in \mathcal{U}_1$$

$$f(x) = \frac{a(x) - A_a}{M+1}$$

$$|f(x)| \le \frac{1}{M+1} (|a(x)| + |A_a|) \le \frac{M}{M+1} < 1.$$

Therefore, $\| f \|_\infty \le 1$ implies $f \in S_\infty$. Hence $\mathcal{U} \subset S_\infty$.

Now for $a \in \mathcal{U}_1$,

$$g * \frac{a - A_a I}{M+1} = \frac{1}{M+1}\left[g * a - A_a \int_{-\infty}^{\infty} g(u)\,du \right] \text{ for every } \frac{a - A_a I}{M+1} \in \mathcal{U}$$

But we know that $g * a(z) \to A_a \int_{-\infty}^{\infty} g(u)du$ as $z \to \infty$ uniformly for $a \in \mathcal{U}_1$, so $\left(g * \frac{a - A_a I}{M+1}\right)(z) \to 0$ as $z \to \infty$ uniformly for $a \in \mathcal{U}_1$

That is $(g * f)(z) \to 0$ as $z \to \infty$ uniformly for $a \in \mathcal{U}_1$

So by the above theorem applied to H & $\mathcal{U}$,

$(h * f)(z) \to 0$ as $z \to \infty$ uniformly for f in $\mathcal{U}$ and h in H ..... (1)

Now for any $h_1 \in H_1$ and $a \in \mathcal{U}_1$, we have

$$\frac{h_1}{\|h_0\|_1 + 1} \in H \text{ and } \frac{a - A_a I}{M+1} \in \mathcal{U}$$

Thus from equation (1) we have

$$\left(\frac{h_1}{\|h_0\|_1 + 1} * \frac{a - A_a I}{M+1}\right)(z) \to 0 \text{ as } z \to \infty \text{ uniformly for } a \in \mathcal{U}_1 \text{ and } h_1 \in H_1.$$

i.e., $(h_1 * (a - A_a I))(z) \to 0$ as $z \to \infty$ uniformly for a in $\mathcal{U}_1$ and h_1 in H_1.

i.e., $(h_1 * a)(z) \to A_a \int_{-\infty}^{\infty} h_1(u)du$, as $z \to \infty$ uniformly for a in $\mathcal{U}_1$ and h_1 in H_1.

This completes the proof.

If we take $\mathbb{R} = G$, a locally compact abelian group with Haar measure μ, the following result can be proved.

Theorem 2.5. *$\mathcal{U} \subset S_\infty$ such that $\{\phi_a : a \in \mathcal{U}\}$ given by $\phi_a(x) = a_x$, $x \in G$ is uniformly equicontinuous from G to $L^\infty(G)$.*

$H \subset L^1(G)$, there exists $h_0 \in S_1$, $|h(t)| \le |h_0(t)|$ for all $h \in H$ and $t \in G$. Let $g \in S_1 \cap \mathsf{U}$. Suppose that if $g * a(x) \to 0$ as $x \to \infty$ in G uniformly for a in $\mathcal{U}$, then $h * a(x) \to 0$ as $x \to \infty$ in G uniformly for a in $\mathcal{U}$ and h in H.

Proof: The proof of this theorem follows from theorem 2.3.

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