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Uniform version of Wiener–Tauberian theorem for Wiener algebra

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Abstract: The Wiener-Tauberian theorem for $\mathbb{R}$ says that the closed translation invariant subspace generated by $f \in L^1(\mathbb{R})$ is $L^1(\mathbb{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we prove a uniform version of this result in Wiener algebra.

Keywords: Wiener-Tauberian theorem, locally compact abelian group, translation invariant subspace, Wiener algebra.

1.Introduction:

Let G be a separable locally compact abelian group with left Haar measure λ. Let K be a nonempty compact subset of G which is the closure of its interior. If f is measurable function on G, we define

$$f^{\#}: G \to [0,\infty] \text{ by}$$
$$f^{\#}(x) = \| f\xi_{xK} \|_{\infty}$$

Let $\mathcal{N}_1$ be the set of Measurable functions g on G for which $g^{\#} \in L_1$. $\mathcal{N}_1$ is a Banach space with norm $\| f \|_1^{\#} = \| f^{\#} \|_1$. In fact $\mathcal{N}_1$ is a Banach algebra under Convolution.

The Wiener algebra $W(G)$ is defined as follows:

$$W(G) = \{ f \in \mathcal{N}_1 \, ; \, f \text{ is continuous} \}$$

Let $\mathscr{R}$ be the set of all Radon measures on G. For $\mu \in \mathscr{R}$ we define

$$\mu^{\#}: G \to [0,\infty] \text{ by}$$
$$\mu^{\#}(x) = \| \xi_{xK}\mu \| = |\mu|(xK)$$

also $\mu^{\#} = |\mu| * \xi_{K^{-1}}$

$$W(G)^* = \{ \mu \in \mathscr{R} \, ; \, \mu^{\#} \in L_{\infty} \} \text{ with the norm } \| \mu \| = \| \mu^{\#} \|_{\infty}$$

$$(f, \psi\mu) = \psi(\mu) f = \int_G f \, d\mu, \; f \in W(G) \, ; \, \mu \in W(G)^*$$

For $h \in \mathscr{H}$, $\Phi_h : G \to L^1(G)$ by

$$\Phi_h(x) = {}_x h, x \in G$$

Let $S_1^{W(G)}$ and $S_\infty^{W(G)^*}$ be the unit ball of $W(G)$ and $W(G)^*$ respectively.

$$\mathrm{U} = \{g \in W(G)\ ; \text{ for } \mu \in \mathscr{R},\ g * \mu = 0 \Rightarrow \mu = 0\ \}$$

For $f \in C_{00}(G)$

$$\begin{aligned}
(1.1) \qquad \| ({}_x f)^{\#} \|_1 &= \int_G |({}_x f)^{\#}(y)|\, d\lambda(y) \\
&= \int_G \| {}_x f \xi_{yK} \|_\infty \, d\lambda(y) \\
&= \int_G \operatorname*{Sup}_{z \in G} | f(xz) | \, \xi_{yK}(z) d\lambda(y) \\
&= \int_G \operatorname*{Sup}_{z \in G} | f(z) | \, \xi_{yK}(x^{-1} z)\, d\lambda(y) \\
&= \int_G \operatorname*{Sup}_{z \in G} | f(z) | \, \xi_{K}(y^{-1} x^{-1} z)\, d\lambda(y) \\
&= \int_G \operatorname*{Sup}_{z \in G} | f(z) | \, \xi_{K}(y^{-1} z)\, d\lambda(y) \\
&= \int_G \operatorname*{Sup}_{z \in G} | f(z) | \, \xi_{yK}(z)\, d\lambda(y) \\
&= \int_G \operatorname*{Sup}_{z \in G} | f \xi_{yK}(z) | \, d\lambda(y) \\
&= \int_G \| f \xi_{yK} \|_\infty \, d\lambda(y) \\
&= \| f \|_{W(G)}
\end{aligned}$$

since $C_{00}(G)$ is dense in $W(G)$,

so $\| {}_x f \|_{W(G)} = \| f \|_{W(G)}$ for all $f \in W(G)$.

Let $h \in C_{00}(G)$,

$$\begin{aligned}
\| h \|_{W(G)} = \| h^{\#} \|_1 &= \int_G | h^{\#}(x) | \, d\lambda(x) \\
&= \int_G \operatorname*{Sup}_{y \in G} \| h \xi_{xK} \|_\infty \, d\lambda(x) \\
&= \int_G \operatorname*{Sup}_{y \in G} | h \xi_{xK}(y) | \, d\lambda(x) \\
&= \int_G \operatorname*{Sup}_{y \in G} | h(y) | \, \xi_{xK}(y)\, d\lambda(x)
\end{aligned}$$

$$\leq \int_G \operatorname{Sup}_{y\in G} |\tilde{h}(y)|\, \xi_{xK}(y)\, d\lambda(x)$$

$$= \int_G |\tilde{h}^{\#}(x)|\, d\lambda(x) = \|\tilde{h}\|_{W(G)}$$

Let us take, $h \in C_{00}(G)$

1.3. $$\int h(x)\tilde{\mu}^{\#}(x)\, d\lambda(x) = \int_G \int_G h(x)\xi_{xK}(y)\, d\,|\widetilde{\sim|}(y)\, d\lambda(x)$$

$$= \int_G \int_G h(x)\xi_K(x^{-1}y^{-1})\, d\,|\mu|(y)\, d\lambda(x)$$

$$= \int_G \int_G h(x)\xi_{K^{-1}}(yx)\, d\,|\mu|(y)\, d\lambda(x)$$

$$= \int_G \int_G \xi_{K^{-1}}(x)\, h(y^{-1}x)\, d\,|\mu|(y)\, d\lambda(x)$$

$$= \int_G \int_G h(y^{-1}x)\xi_K(x^{-1})\, d\lambda(x)\, d\,|\mu|(y)$$

$$= \int_G (h * \xi_K)(y^{-1})\, d\,|\mu|(y)$$

$$= \langle \xi_{K^{-1}} * \tilde{h}, |\mu| \rangle$$

and hence $\tilde{\mu}^{\#} \in L_\infty$.

1.4. For $f \in W(G)$ and $\mu \in W(G)^*$

$$\mu * f(x) = \int_G f(y^{-1}x)\, d\mu(y)$$

$$= \int_G f(xy^{-1})\, d\mu(y)$$

$$= \int_G {}_x f(y^{-1})\, d\mu(y)$$

$$= \int_G {}_x f(y)\, d\mu(y^{-1})$$

$$= \int_G {}_x f(y)\, d\tilde{\mu}(y)$$

$$= \psi(\tilde{\mu})({}_x f)$$

1.5. Theorem : *Let G be a separable locally compact abelian group. Let $\mathcal{H} \subset W(G)$ be such that the family $\{\Phi_h : h \in \mathcal{H}\}$ is uniformly equicontinuous. Suppose there exist $\tilde{h} \in \mathcal{H}$ with $|h(t)| \leq |\tilde{h}(t)|$ and $\|h\|_{W(G)} \leq \|\tilde{h}\|_{W(G)}$ for all $h \in \mathcal{H}$ & $t \in G$.*

Let $g \in U$ be fixed and $\mathcal{U} \subset S_\infty^{W(G)^*}$. Suppose that $\mu * g(x) \to 0$ as $x \to \infty$ uniformly for μ in $\mathcal{U}$ then $h * \mu(x) \to 0$ as $x \to \infty$ uniformly for h in $\mathcal{H}$ and μ in $\mathcal{U}$.

Proof: Assume to the contrary. So there exists $\delta > 0$ such that for every compact set K in G there exists $x_K \in G \sim k$, $h_K \in \mathscr{H}$ and $h_K \in \mu_K \in \mathcal{U}$ satisfying :

$$|(h_K * \mu_K)(x_K)| > \delta.$$

Since G is separable and locally compact so G is σ-compact. Thus there exists an increasing sequence say $\{K_n\}_{n\in\mathbb{N}}$ of compact sets with $K_n \subset \text{Int } K_{n+1}$ and if F is any compact set in G thus there exists n_0 with $F \subset K_{n_0}$. We write

$$h_{K_n} = h_n, \mu_{K_n} = \mu_n \text{ and } x_{K_n} = x_n.$$

Define a sequence of functions on G by

$$\Delta_n(x) = {}_{x_n}(h_n * \mu_n)(x)$$

$$\begin{aligned}
\sup_{x\in G} |\Delta_n(x)| &= \sup_{x\in G} |{}_{x_n}(h_n * \mu_n)(x)| \\
&= \sup_{x\in G} |(h_n * \mu_n)(x_n x)| \\
&= \sup_{x\in G} |\psi(\bar{\mu}_n)({}_{x_n x}(h_n))| \\
&\leq \|\psi(\bar{\mu}_n)\|_{W(G)^*} \|{}_{x_n x} h_n\|_{W(G)} \\
&= \|\bar{\mu}_n\|_{W_\infty} \|h_n\|_{W(G)} \\
&= \|\bar{\mu}n^{\#}\|_\infty \|\bar{h}\|_{W(G)} \\
&\leq 1
\end{aligned}$$

Since $\{\Phi_h : h \in \mathscr{H}\}$ is uniformly equi continuous, so for given $\in > 0$ there exists a nbd $U_\in$ of e in G s.t. for $t \in U_\in$, for $x \in G$, $y = tx$ we have

$$\left\| {}_x h - {}_y h \right\|_{W(G)} < \in.$$

For $y = tx$, $t \in U_\in$.

$$\begin{aligned}
|\Delta_n(x) - \Delta_n(y)| &= |{}_{x_n}(h_n * \mu_n)(x) - {}_{x_n}(h_n * \mu_n)(y)| \\
&= |(h_n * \mu_n)(x_n x) - (h_n * \mu_n)(x_n y)| \\
&= |\psi(\bar{\mu}_n)({}_{x_n x}(h_n)) - \psi(\bar{\mu}_n)({}_{x_n y}(h_n))| \\
&= |\psi(\bar{\mu}_n)({}_{x_n x}(h_n)) - {}_{x_n y}(h_n)| \\
&= |\psi(\bar{\mu}_n)\{{}_{x_n}({}_x(h_n) - {}_y(h_n))\}| \\
&\leq \|\psi(\bar{\mu}_n)\|_{W(G)^*} \|{}_{xn}({}_x(h_n) - {}_y(h_n))\|_{W(G)}
\end{aligned}$$

$$= \| \tilde{\mu}_n \|_{W_\infty} \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$= \| \tilde{\mu}_n^{\#} \|_{\infty} \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$\leq \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$< \epsilon$$

By Ascoli's theorem [5] there exists a pointwise Convergent subsequence $\{\Delta_{n_j}\}$ converging to a continuous function Δ on G. Thus for a fixed x and t in G.

$$\Delta_{n_j}(xt^{-1}) \to \Delta(xt^{-1}),\ j \to \infty$$
$$\left| \Delta_{n_j}(xt^{-1})\, g(t) \right| = \left| \Delta_{n_j}(xt^{-1}) \right| |g(t)|$$
$$\leq \| \Delta_{n_j} \|_\infty |g(t)|.$$
$$\leq |g(t)|$$

Thus by Lebesgue dominated Convergence theorem

$$\int_G \Delta_{n_j}(xt^{-1})\, g(t)\, d\lambda(t) \to \int_G \Delta(xt^{-1})\, g(t)\, d\lambda(t)$$
$$\text{i.e. } \Delta_{n_j} * g(x) \to s * g(x)\ \ \forall x \in G$$

Now,
$$\Delta_{n_j} * g(x) = \int_G \Delta_{n_j}(xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \int_G x_{n_j}(h_{n_j} * \mu_{n_j})(xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \int_G (h_{n_j} * \mu_{n_j})(x_{n_j} xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \left((h_{n_j} * \mu_{n_j}) * g \right)(x_{n_j} x)$$
$$= \left((h_{n_j} * (\mu_{n_j} * g) \right)(x_{n_j} x)$$
$$= (\mu_{n_j} * g) * h_{n_j}(\mu_{n_j} x)$$
$$= \int_G \cup_{n_j}^{x}(y)\, d\lambda(y)$$

Where $\cup_{n_j}^{x}(y) = h_{n_j}(y)(\mu_{n_j} * g)(y^{-1} x_{n_j} x)$. Since $\mu * g$ vanishes at infinity uniformly for $\mu \in \mathcal{U}$. We have for any $n \in \mathbb{N}$ there is a compact set $\tilde{K}_n$ s.t $|\mu * g(z)| < \frac{1}{n}$ for $z \in G \sim \tilde{K}_n$ and $\mu \in \mathcal{U}$.

For $y \in G$, let $L_n^{x,y} = y \tilde{K}_n x^{-1}$ which is compact so there exists $j_n^{x,y}$ such that $L_n^{x,y} \subset K_j$ for $j \geq j_n^{x,y}$ and as $x_{n_j} \notin K_j$, we have,

$$= \| \bar{\mu}_n \|_{W_\infty} \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$= \| \bar{\mu}_n^{\#} \|_\infty \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$\le \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$< \epsilon$$

By Ascoli's theorem [5] there exists a pointwise Convergent subsequence $\{ \delta_{n_j} \}$ converging to a continuous function δ on G. Thus for a fixed x and t in G.

$$\delta_{n_j}(xt^{-1}) \to \delta(xt^{-1}), \; j \to \infty$$
$$\left| \delta_{n_j}(xt^{-1})\, g(t) \right| = \left| \delta_{n_j}(xt^{-1}) \right| |g(t)|$$
$$\le \| \delta_{n_j} \|_\infty \; |g(t)|.$$
$$\le |g(t)|$$

Thus by Lebesgue dominated Convergence theorem

$$\int_G \delta_{n_j}(xt^{-1})\, g(t)\, d\lambda(t) \to \int_G \delta(xt^{-1})\, g(t)\, d\lambda(t)$$
$$\text{i.e. } \delta_{n_j} * g(x) \to s * g(x) \;\; \forall x \in G$$

Now,
$$\delta_{n_j} * g(x) = \int_G \delta_{n_j}(xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \int_G {}_{x_{n_j}}(h_{n_j} * \mu_{n_j})(xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \int_G (h_{n_j} \cdot \mu_{n_j})(x_{n_j} xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \left((h_{n_j} \cdot \mu_{n_j}) * g \right)(x_{n_j} x)$$
$$= \left((h_{n_j} \cdot (\mu_{n_j} * g) \right)(x_{n_j} x)$$
$$= (\mu_{n_j} * g) * h_{n_j}(\mu_{n_j} x)$$
$$= \int_G \bigcup_{n_j}^{x}(y)\, d\lambda(y)$$

Where $\bigcup_{n_j}^{x}(y) = h_{n_j}(y)(\mu_{n_j} * g)(y^{-1} x_{n_j} x)$. Since $\mu * g$ vanishes at infinity uniformly for $\mu \in \mathcal{U}$. We have for any $n \in \mathbb{N}$ there is a compact set $\tilde{K}_n$ s.t $|\mu * g(z)| < 1/n$ for $z \in G - \tilde{K}_n$ and $\mu \in \mathcal{U}$.

For $y \in G$, let $L_n^{x,y} = y \tilde{K}_n x^{-1}$ which is compact so there exists $j_n^{x,y}$ such that $L_n^{x,y} \subset K_j$ for $j \ge j_n^{x,y}$ and as $x_{n_j} \notin K_j$, we have,

$$\left|(\mu_{n_j} * g)(y^{-1}x_{n_j}x)\right| < 1/n$$

$$|\mathcal{U}n_j^x(y)| \leq |\bar{h}(y)|\, \mu_{n_j} * g(y^{-1}x_{n_j}x)$$
$$\leq |(\bar{h}(y)|\, 1/n\,,\ \ j \geq j_n^{x,y}$$
$$\Rightarrow \cup_{n_j}^x(y) \to 0 \text{ as } j \to \infty$$

Thus by Lebesgue dominated convergence theorem $\int_G u_{n_j}^x(y)\,d\lambda(y) \to 0$ as $j \to \infty$

$$\text{i.e. } \mathcal{S}_{n_j} * g(x) \to 0 \text{ i.e. } \mathcal{S} * g(x) \to 0$$
$$\Rightarrow \mathcal{S} = 0 \text{ since } g \in U.$$

But $\mathcal{S}_n(e) = (h_n * \mu_n)(x_n)$ so $|\mathcal{S}_n(e)| > \delta$

$\Rightarrow |\mathcal{S}(e)| \geq \delta$. Which is contradiction.

This completes the proof.

Let S_∞ be the unit ball of $W(G)^*$ and S_1 be the unit ball of $W(G)$ and $U = \{g : G \to \mathbb{C}$ measurable, $g \in W(G)$; for $a \in B \cap C$, $a * g = 0 \Rightarrow a = 0\}$.

Where B and C are sets of bounded and continuous functions respectively. The following theroem may also holds.

1.6. Theorem: *Let $\mathcal{U} \subset S_\infty$ be such that the family $\{\phi_\mu : \mu \in \mathcal{U}\}$ is uniformly equicontinuous. Let $\mathcal{H} \subset W(G)$, suppose that there exist $\bar{h} \in S_1$ s.t $|h(t)| \leq |\bar{h}(t)|$ for all $h \in \mathcal{H}$ and all $t \in G$. If $g \in S_1 \cap U$ and $\mu^\# * g$ vanishes at infinity uniformly for $\mu \in \mathcal{U}$* then *$\mu^\# * h$ vanishes at infinity uniformly for every $\mu \in \mathcal{U}$ and $h \in \mathcal{H}$.*

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