The Nepali Math. Sc. Report
Vol. 28. No. 1 & 2. 2008 (Dec.)

Uniform version of Wiener–Tauberian theorem for Wiener algebra on a real line

CHET RAJ BHATTA

Central Department of Mathematics
Tribhuvan University, Kirtipur, Kathmandu, Nepal
Email: crbhatta@yahoo.com

Abstract: The Winner-Tauberian theorem for $\mathbf{R}$ says that the closed translation invariant subspace generated by $f \in L^1(\mathbf{R})$ is $L^1(\mathbf{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we prove a uniform version of this result in Winer algebra for real line.

Key words: Weiner-Tauberian theorem, translation invariant subspace, Wiener algebra, Radon measure.

1. Introduction:

For $k = 0, \pm 1, \pm 2, \ldots$, let I_k denotes the closed interval $I_k = [k, k+1]$. Let $W(\mathbf{R})$ be the linear space of all complex valued functions f on $\mathbb{R}$ for which

$$\|f\|_{W(\mathbf{R})} = \sum_{k \in \mathbf{Z}} \max_{x \in I_k} |f(x)| \text{ is finite.}$$

Let $C_{00}(\mathbf{R})$ denotes the set of all complex-valued continuous functions f with compact support. Let $\mathscr{R}$ be the set of Radon measure on $\mathbb{R}$. Then $\mathscr{R}$ can be identified with a sequence $(\mu_n)_{n \in \mathbf{Z}}$, where for $n \in \mathbb{N}$. μ_n is a measure in $M([n, n+1))$. That is, for a Borel set $\mathscr{B}$, let $\mathscr{B}_n = \mathscr{B} \cap [n, n+1)$ so that $\mathscr{B} = \bigcup_{n \in \mathbf{Z}} \mathscr{B}_n$; $\mu B = \sum_{n \in \mathbf{Z}} \mu_n(B_n)$.

In other words, for $f \in C_{00}(\mathbf{R})$,

$$\mu(f) = \int_{\mathbf{R}} f d\mu = \sum_{n \in \mathbf{Z}} \int_{J_n} f d\mu_n = \sum_{n \in \mathbf{Z}} \mu_n (f / J_n)$$

where, $J_n = [n, n+1)$.

Let $\mathcal{R}^b = \{\mu = (\mu_n)_{n\in\mathbb{Z}}$ and $\|\mu\|_b = \sup\{\|\mu_n\| : n\in\mathbb{Z}\}\} < \infty$ is isomorphic with $W(\mathbb{R})^*$ via $\mu\in\mathcal{R}^b \leftrightarrow F_\mu \in W(\mathbb{R})^*$ and given by

$$F_\mu(f) = \int_{\mathbb{R}} f d\mu, \ \ f\in W(\mathbb{R})$$

The norm of the fuctional, which we write as $\|F_\mu\|$, satisfies the following inequalities:

$$\tfrac{1}{2}\|\mu\|_b \le \|F_\mu\| \le \|\mu\|_b$$

Let $\mathrm{U} = \{g\in W(\mathbb{R}) : \text{for } \mu\in\mathcal{R},\ g*\mu = 0 \Rightarrow \mu = 0\}$

Let W_1 be the unit ball of $W(\mathbb{R})$ and $\mathcal{R}_1^\infty$ be the unit ball of $\mathcal{R}^b$ i.e.

$\mathcal{R}_1^\infty = \{\mu\in\mathcal{R}^b : \|\mu\|_b \le 1\}$

For $f\in W(\mathbb{R})$ and $\mu\in\mathcal{R}^b$, the convolution is defined by,

$$f*\mu(x) = \int_{\mathbb{R}} f(x-y)\, d\mu(y)$$

and
$$\begin{aligned}\int_{\mathbb{R}} f(x-y)\, d\mu(y) &= \int_{\mathbb{R}} f_x(-y)\, d\mu(y) \\ & \qquad y\to -y \\ &= \int_{\mathbb{R}} f_x(y)\, d\mu(-y) \\ &= \int_{\mathbb{R}} f_x(y)\, d\tilde{\mu}(y) \ \ (\because\ \tilde{\mu}(y) = \mu(-y)) \\ &= F_{\tilde{\mu}}(f_x)\end{aligned}$$

Theorem: *Let $\mathcal{H}\in W(\mathbb{R})$ be such that*

(i) $\{\mathcal{S}_h : h\in\mathcal{H}\}$ *given by* $\mathcal{S}_h(x) = h_x, x\in\mathbb{R}$ *is uniformly equicontinuous.*

(ii) $\exists \tilde{h}\in W_1$ *with* $|h(t)| \le |\tilde{h}(t)|\ \forall\ h\in\mathcal{H}$ *and* $\forall\ t\in\mathbb{R}$.

Let $g\in W_1\cap U$. Let $\mathcal{U}\subset\mathcal{R}_1^\infty$. Suppose that $g*\mu(x)\to 0$ as $x\to\infty$ uniformly for μ in $\mathcal{U}$ then $h*\mu(x)\to 0$ as $x\to\infty$ uniformly for h in $\mathcal{H}$ and μ in $\mathcal{U}$.

Proof: Assume to the contrary that there exists $\delta > 0$ such that $\forall n$ there exists $x_n\in\mathbb{R}$ with $x_n > n$, $h_n\in\mathcal{H}$ and $\mu_{(n)}\in\mathcal{U}$ satisfying:

$$\left|\left(h_n * \mu_{(n)}\right)(x_n)\right| > \delta.$$

Let us consider the sequences

$$\mathcal{s}_n(x) = \left(h_n * \mu_{(n)}\right)_{x_n}(x),\ n = 1,2,3,\ \ldots,\ x \in \mathbb{R}$$
$$= \left(h_n * \mu_{(n)}\right)(x + x_n)$$

$\mathcal{s}_n$ is measurable. Now we shall show that it is bounded and equicontinuous on $\mathbb{R}$.

$$\sup_{x\in\mathbb{R}} |s_n(x)| = \sup_{x\in\mathbb{R}} \left| \left(h_n * \mu_{(n)}\right)_{x_n}(x) \right|$$
$$= \sup_{x\in\mathbb{R}} \left| \left(h_n * \mu_{(n)}\right)(x + x_n) \right|$$
$$= \sup_{x\in\mathbb{R}} \left| F_{\bar{\mu}_{(n)}}\left((h_n)_{x+x_n}\right) \right|$$
$$\leq \left\| F_{\bar{\mu}_{(n)}} \right\|_{W(\mathbb{R})^*} \left\| (h_n)_{x+x_n} \right\|_{W(\mathbb{R})}$$
$$\leq 2 \left\| F_{\bar{\mu}_{(n)}} \right\|_b \left\| \bar{h} \right\|_{W(\mathbb{R})}$$
$$\leq 2$$

$\therefore\ \{\mathcal{s}_n\}_{n\in\mathbf{N}}$ is uniformly bounded.

Since, $\{\mathcal{S}_n : h \in \mathcal{H}\} = \tau$ is uniformly equicontinuous, so for a given $\frac{\epsilon}{2} > 0$ there corresponds a $\delta > 0$ such that $\left\| h_x - h_y \right\|_{W(\mathbb{R})} = \|\mathcal{S}_h(x) - \mathcal{S}_h(y)\|_{W(\mathbb{R})} < \epsilon/2, \forall\ \mathcal{S}_h \in \tau$ and for all pair of point x, y with $|x - y| < \delta$.

For $x, y \in \mathbb{R}$,

$$|\mathcal{s}_n(x) - \mathcal{s}_n(y)| = \left| \left(h_n * \mu_{(n)}\right)_{x_n}(x) - \left(h_n * \mu_{(n)}\right)_{x_n}(y) \right|$$
$$= \left| \left(h_n * \mu_{(n)}\right)(x + x_n) - \left(h_n * \mu_{(n)}\right)(y + x_n) \right|$$
$$= \left| \left(F_{\bar{\mu}_{(n)}}\left(h_n\right)_{x+x_n} \right) - F_{\bar{\mu}_{(n)}}\left((h_n)_{y+x_n}\right) \right|$$
$$= \left| F_{\bar{\mu}_{(n)}} \left\{ \left((h_n)_x\right)_{x_n} - \left((h_n)_y\right)_{x_n} \right\} \right|$$
$$= \left| F_{\bar{\mu}_{(n)}} \left\{ \left((h_n)_x - (h_n)_y\right)_{x_n} \right\} \right|$$
$$\leq \left\| F_{\bar{\mu}_{(n)}} \right\|_{W(\mathbb{R})^*} \left\| \left((h_n)_x - (h_n)_y\right)_{x_n} \right\|_{W(\mathbb{R})}$$
$$\leq 2 \left\| F_{\bar{\mu}_{(n)}} \right\|_{W(\mathbb{R})^*} \left\| (h_n)_x - (h_n)_y \right\|_{W(\mathbb{R})}$$
$$\leq 2 \left\| F_{\bar{\mu}_{(n)}} \right\|_b \left\| (h_n)_x - (h_n)_y \right\|_{W(\mathbb{R})}$$
$$\leq 2 \left\| (h_n)_x - (h_n)_y \right\|_{W(\mathbb{R})}$$
$$< \epsilon.$$

and therefore, $|\mathcal{S}_n(x)-\mathcal{S}_n(y)|<\epsilon$ for $n\in\mathbb{N}$ and $|x-y|<\delta$. Therefore, $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous on $\mathbb{R}$.

Thus by Ascoli's Lemma [5] we now select from the sequence $\mathcal{S}_n(x)$ a subsequences $\mathcal{S}_{n_k}(x)$ which tends to limit $\mathcal{S}(x)$ pointwise as $k\to\infty$ and continuous on $\mathbb{R}$. For each fixed $x\in\mathbb{R}$ and $t\in\mathbb{R}$, $\mathcal{S}_{n_k}(x-t)\to\mathcal{S}(x-t)$, $k\to\infty$ and therefore, $\mathcal{S}_{n_k}(x-t)g(t)\to\mathcal{S}(x-t)g(t)$, $k\to\infty$.

Now $\forall t\in\mathbb{R}$,

$$\begin{aligned}|\mathcal{S}_{n_k}(x-t)g(t)| &= |\mathcal{S}_{n_k}(x-t)\ g(t)|\\ &\le \|\mathcal{S}_{n_k}\|_\infty\,|g(t)|\\ &\le 2|g(t)|\end{aligned}$$

Thus by Lebesgue dominated convergence therems, $\forall x\in\mathbb{R}$

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{S}_{n_k}(x-t)\,g(t)\,dt \to \int_{\mathbb{R}}\mathcal{S}(x-t)\,g(t)\,dt,\ \ k\to\infty\\ \equiv(\mathcal{S}*g)(x)\end{aligned}$$

Now,

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{S}_{n_k}(x-t)\,g(t)\,dt &= \int_{\mathbb{R}}\left(h_{n_k}*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)g(t)\,dt\\ &= \left(h_{n_k}*\mu_{(n_k)}*g\right)\left(x_{n_k}+x\right)\\ &= \left(\left(g*\mu_{(n_k)}\right)*h_{n_k}\right)\left(x_{n_k}+x\right)\\ &= \int_{\mathbb{R}}\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)h_{n_k}(t)\,dt\end{aligned}$$

Put $I_{k,x}(t)=\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)h_{n_k}(t)$

Since we know, $(g*\mu)(z)\to 0$ as $z\to\infty$ uniformly for μ in $\mathcal{U}$. we have for given $\epsilon>\exists\Delta : |(g*\mu)(z)|<\epsilon\ \forall z\ge\Delta$ and μ in $\mathcal{U}$.

Therefore $\left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\right|<\epsilon$ for $x_{n_k}+x-t\ge\Delta$

Thus for a fixed x and t in $\mathbb{R}$,

$$\left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\right|<\epsilon \ \text{ for } x_{n_k}+x-t\ge\Delta.$$

Thus for a fixed x and t in $\mathbb{R}$,

$$\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\to 0 \text{ as } k\to\infty$$

$$\begin{aligned}\left|I_{k,x}(t)\right| &= \left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)h_{n_k}(t)\right| \\ &= \left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\right|\left|h_{n_k}(t)\right| \\ &\le \in \left|h_{n_k}(t)\right| \\ &\le \in \left|\tilde{h}(t)\right|\end{aligned}$$

Thus by applying Lebesgue dominated convergence theorem for each $x\in\mathbb{R}$,

$$\int_{\mathbb{R}} I_{k,x}(t)\,dt\to 0 \text{ as } k\to\infty$$

and therefore $\Delta * g(x)=0$. Since $g\in U$ we get $\Delta=0$.

But

$$\begin{aligned}|\Delta(0)| &= \lim_{k\to\infty}\left|\Delta_{n_k}(0)\right| \\ &= \lim_{k\to\infty}\left|\left(h_{n_k}*\mu_{(n_k)}\right)\left(x_{n_k}\right)\right| \\ &\ge \delta>0\end{aligned}$$

which is a contradiction and therefore

$$\int_{\mathbb{R}} h(x-t)d\mu(t)\to 0 \text{ as } x\to\infty \text{ uniformly for } h \text{ is } \mathscr{H} \text{ and } \mu \text{ in } \mathscr{U},$$

Example: Take $f(x)=e^{-|x|}$

$$\begin{aligned}\hat{f}(y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\exp(-iyx)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(-|x|)\exp(-iyx)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}\exp(x)\exp(-iyx)\,dx+\int_{0}^{\infty}\exp(-x)\exp(-iyx)\,dx\right. \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}\exp\left((1-iy)x\right)dx+\int_{0}^{\infty}\exp\left(-(1+iy)x\right)dx\right]\end{aligned}$$

Putting $x\to -x$ in first integral

$$= \frac{1}{\sqrt{2\pi}} \Big[\int_0^{\infty} \exp\big(-(1-iy)\,x\big)\,dx + \int_0^{\infty} \exp\big(-(1+iy)\,x\big)\,dx \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\int_0^{\infty} \exp(-x)\,\big[\exp(iyx) + \exp(-iyx)\big]\,dx \Big]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+y^2}$$

$\hat{f}(y) \to 0$ as $y \to \pm\infty$.

f is continuous function on $\mathbb{R}$ and so is measurable

$$\sum_{n=-\infty}^{\infty} \max_{x\in[n,n+1]} e^{-|x|} = \sum_{n=-\infty}^{0} \max_{x\in[n,n+1]} e^{-|x|} + \sum_{n=0}^{\infty} \max_{x\in[n,n+1]} e^{-|x|}$$

$$= \sum_{n=-\infty}^{0} e^{-|n+1|} + \sum_{n=0}^{\infty} e^{-|n|}$$

put $n+1=-k,\ k>0$

$$= \sum_{k=\infty}^{0} e^{-|-k|} + \sum_{n=0}^{\infty} e^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-n} + \sum_{n=0}^{\infty} e^{-n}$$

$$= 2\sum_{n=0}^{\infty} e^{-n}$$

$$= 2\sum_{n=0}^{\infty} \frac{1}{e^n} = \frac{2e}{1-e} < \infty$$

therefore $e^{-|x|} \in W(\mathbb{R})$

since $$\|f\|_{W(\mathbb{R})} = \sum_{n\in\mathbb{Z}} \max_{x\in[n,n+1]} |f(x)|$$

$$= \sum_{n\in\mathbb{Z}}^{\infty} \|f\|_{[n,n+1]}$$

But $$\sum_{n\in\mathbb{N}} \|f\|_{[n,n+1]} \le \sum_{n\in\mathbb{Z}} \|f\|_{[n,n+1]} < \infty$$

Thus applying the definition of convergence on $\mathbb{N}$. Let $\epsilon > 0$ then $n_0 \in \mathbb{N}$. Such that

$$T_{n_o} = \sum_{n \geq n_o} \|f\|_{[n,n+1]} \to 0 \text{ as } n_o \to \infty.$$

$[t] =$ The greater integer $\leq t$

$$T_{[t]} = \sum_{n \geq [t]} \|f\|_{[n,n+1]} = \sum_{n \geq t} \|f\|_{[n,n+1]}$$

$$\mu(f) = \sum_{n \in \mathbf{N}} f(n)$$

$$(\mu * f)(t) = \mu(\tilde{f}_t) = \sum_{n \in \mathbf{N}} f(t+n) = \sum_{n \in \mathbf{N}} e^{-|t+n|}$$

$$= e^{-|t+1|} + e^{-|t+2|} + \ . \ . \ .$$

But $\quad |f(t)| = e^{-|t|} \leq \|f\|_{([t],[t]+1)}$

$$|f(t+1)| = e^{-|t+1|} \leq \|f\|_{([t]+1,[t]+2)}$$

$$|f(t+2)| = e^{-|t+2|} \leq \|f\|_{([t]+2,[t]+3)} \quad \text{and so on.}$$

Therefore $\quad |f(t)| + \sum_{n \in \mathbf{N}} |f(t+1)| \leq \sum_{n \geq [t]} \|f\|_{[n,n+1]} = T_{[t]} \to 0 \quad t \to \infty.$

$\therefore\ (\mu * f)(t) \to 0$ as $t \to \infty$.

Now, $\quad (\mu * f_s)(t) = \sum_{n \in \mathbf{N}} f(s+t+n) = \sum_{n \in \mathbf{N}} e^{-|s+t+n|}$ take $s = -t$

and therefore,

$$\mu * f_s(t) = \sum_{n \in \mathbf{N}} e^{-n} < \infty$$

$\therefore\ (\mu * f_s)(t) \nrightarrow 0$ as $t \to \infty$.

REFERENCES

[1] Benedetto, J. J., (1975), *Spectral Synthesis*, B. G. Teubnersttutgart.

[2] Bhatta, C. R., (2004), Uniform version of Wiener Tauberian theorem for real line, *The Nepali Mathematical Sciences Report*, Vol. **23**(2), 9–16.

[3] Bhatta, C. R., (2006), Uniform Version of Wiener-Tauberian theorem for equicontinuous subsets of subspace of $L^1(X,\mu)$, *The Nepali Mathematical Sciences Report*, Vol. **26**(1&2), 19–26.

[4] Hewitt, E., and Ross, K. A., (1963–1970), *Abstract Harmonic analysis* I & II, Springer Verlag.

[5] Kelley, J. L., (1961), *General Topology*, D. Van Nostrand Company, Inc.

[6] Kumar, A., and Bhatta, C. R., (2003), A uniform version of Wiener-Tauberian theorem, *Journal of Mathematical Sciences*, Vol. 2. 63–71.

[7] Reiter, H., and Stegeman, J. D., (2000), *Classical harmonic analysis and locally compact groups*, Oxford University Press.

[8] Wiener, N., (1933), *The Fourier integral and certain of its applications*, Cambridge, England, Cambridge University Press, Reprinted by Dover publ., New York.