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Wiener Tauberian Theorem for Locally Compact Abelian Group

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Abstract:

If $X = \Re^+$, then X is locally compact abelian group under multiplication. Our aim is to prove uniform version of the wiener Tuberian Theorem for $\Re^+$ under new product.

Key words: Winer-Tuberian Theorem, Locally compact abelian group, Translation invariant subspace.

1. Introduction

Wiener in his attempt to characterize rigorously the spectral analysis of a white light signal $\phi \in L^\infty(\Re)$ had to introduce Tuberian arguments, an interesting description of which can be found in Wieners classical work ([4], [5], [6])

The illustration is

Theorem 1.1:

Let $f \in L^1(\Re)$ have a non vanishing Fourier transform $\hat{f}$ and $\phi \in L^\infty(\Re)$. Assume that $\int_\Re |xf(x)|\ dx < \infty$ and $\lim\limits_{x \to \infty} f*\phi(x) = r\int_\Re f(y)dy$ *where* $*$ denotes the convolution then for each $g \in L^1(\Re)$ $\lim\limits_{x \to \infty} g * \phi(x) = r\int_\Re g(y)\,dy$.

The general Tauberian theorem proved by N. Weiner [6] says that if $g \in L^1(\Re)$is a uniqueness function in the sense that its Fourier transform $\hat{g}$ vanishes nowhere on $\Re$ (and thus the closed translation invariant subspace generated by g is $L^1(\Re)$ and $\phi \in L^\infty(\Re)$ is such that $g * \phi(\underline{x}) \to A\ \hat{g}(o)$ (A is a complex number) as $x \to \infty$ then for every $f \in L^1(\Re)$, $f * \phi\ (x) \to A\ \hat{f}\ (o)$ as $x \to \infty$.

We may take the uniqueness function g as

(i) $g(x) = \exp.(-\alpha\ |x|)$ where α is +ve constant then

$$\hat{g}(y) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\alpha}{\alpha^2 + y^2}$$

(ii) For $\alpha > 0$, let g_a be the function on $\Re$ defined by $g_a(x) = \left(1 - \frac{|x|}{a}\right) \xi_{[-a,\, a]}\ (x)$

Then,

$$\hat{g}_a(y) = (2\pi)^{-1/2}\, \alpha \left[\frac{\sin \frac{1}{2}\alpha y}{\frac{1}{2}\alpha y}\right]^2 \quad \text{for } y \neq 0, \text{ and } \hat{g}_a(o) = (2\pi)^{-1/2}\, \alpha$$

2. Uniform Version

Let G be a locally compact group with left Haar measure μ and Δ be the modular function on G. For $x \in G$ and $f \in L^p(G)$, $1 \le p \le \infty$, let ${}_xf$ and f_x be the left and right translates of f. Let $\phi_f : G \to L^\infty$ (G) be defined by $\phi_f(x) = f_x$, $x \in G$, $f \in L^\infty(G)$. We denote by S_1 and S_∞, the unit balls of $L_1(G)$ and $L^\infty(G)$ respectively. Let Y be the set of all bounded continuous function on G.

Define U = $\{g : G \to \mathbb{C}\ |g$ is measurable function such that $\left(\frac{1}{\Delta}\right) g \in S_1$ and for $a \in Y, a * g = 0 \Rightarrow a = 0\}$.

In this paper we are going to prove uniform version of Weiner Tayberian theorem for $X = \Re^+$ under multiplication and new convolution product as used in the technique of following theorem.

Theorem 2.1:

Let G be a separable locally compact group. For $\mathcal{H} \subset L^1(G)$, suppose that there exists $h_0 \in S_1$ such that $|h(t)| \leq |h_0(t)|$ for all $h \in \mathcal{H}$ and $t \in G$. Let $\boldsymbol{u} \subset S_\infty$ be such that the family $\{\phi_a : a \in \boldsymbol{u}\}$ is right uniformly equicontinuous. If $g \in U$ and $a * g(x) \to 0$ as $x \to \infty$ uniformly for $a \in \boldsymbol{u}$ then $h * a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \boldsymbol{u}$ and $h \in \mathcal{H}$.

Remark 2.2:

If $X = \Re^+$, then X is locally compact abelian group under multiplication, let μ be the Lebesgue measure on X. For $g \in L^1(X)$ and $\psi \in L^\infty(X)$, the product given by $g \odot \psi(x) = \frac{1}{x} \int_X g(t/x)\ \psi(t)\ d\mu(t)$ is not a convolution product but the uniform version of the Wiener Tauberian theorem can be proved for this product also using the techniques of theorem 2.1. However we need to prove the followings:

(i) $\| g_x \|_1 = \frac{1}{x} \|g\|_1$ for $x \in X$ and $g \in L^1(X)$

(ii) ${}_y(g \odot \psi) = g \odot {}_y\psi$

(iii) ${}_yg \odot \psi = \left(\frac{1}{y}\right) g \odot {}_{1/y}\psi$

(iv) $\|g \odot \psi\|_\infty \leq \|g\|_1\, \|\psi\|_\infty$

Proof:

(i) $$\| g_x \|_1 = \int_X |g_x(y)| d\mu(y) = \int_X |g(yx)|\, d\mu(y)$$

$$= \int_X |g(t)| \frac{d\mu(t)}{x}$$

$$= \frac{1}{x} \|g\|_1$$

(ii) $${}_y(g \odot \psi)(x) = g \odot \psi(yx) = \frac{1}{yx} \int_X g\left(\frac{t}{yx}\right) \psi(t)\, d\mu(t)$$

$$= \frac{1}{yx}\int_X g\left(\frac{t'}{x}\right) \psi(t'y)\, y\, d\mu(t')$$

$$= \frac{1}{x}\int_X g\left(\frac{t'}{x}\right)_y \psi(t')\, d\mu(t')$$

$$= (g \odot {}_y\psi)(x)$$

(iii) $$({}_y g \odot \psi)(x) = \frac{1}{x}\int_X {}_y g\left(\frac{t}{x}\right) \psi(t)\, d\mu(t)$$

$$= \frac{1}{x}\int_X g\left(\frac{yt}{x}\right) \psi(t)\, d\mu(t)$$

$$= \frac{1}{x}\int_X g\left(\frac{t'}{x}\right) \psi\left(\frac{t'}{y}\right) \frac{1}{y}\, d\mu(t')$$

$$= \frac{1}{yx}\int_X g\left(\frac{t'}{x}\right) ({}_{1/y}\psi)(t')\, d\mu(t')$$

$$= \frac{1}{y}\, g \odot {}_{1/y}\psi(x)$$

(iv) $$\|g \odot \psi\|_\infty = \sup_{x\in X} |g \odot \psi(x)| = \sup_{x\in X} \left|\frac{1}{x}\int_x g\left(\frac{t}{x}\right) \psi(t)\, d\mu(t)\right|$$

$$\leq \sup_{x\in X}\int_X g\left(\frac{t}{x}\right) |\psi(t)|\, \frac{d\mu(t)}{x} \leq \|g\|_1 \|\psi\|_\infty.$$

Let $d\mu(x)$ be the usual Lebesgue measure on X.

Main Result:

Theorem 2.3:

Let X be locally compact abelian group under multiplication. For $\mathcal{H} \subset L^1(X)$, suppose that there exist $h_o \in S_1$ such that $|h(t)| \leq |h_o(t)|$ for all $h \in \mathcal{H}$ and $t \in G$. Let $\mathcal{U} \subset S_\infty$ be such that the family $\{\phi_\psi : \psi \in \mathcal{U}\}$ is uniformly equicontinuous from X to $L^\infty(X)$: If $g \in U$ and $\frac{1}{x}\int_X g\left(\frac{t}{x}\right) \psi(t)\, d\mu(t) \to 0$ as $x \to \infty$ uniformly for ψ in $\mathcal{U}$ then $\frac{1}{x}\int_X h\left(\frac{t}{x}\right) \psi(t)\, d\mu(t) \to 0$ as $x \to \infty$ uniformly for h in $\mathcal{H}$ and ψ in $\mathcal{U}$.

Proof:

Assuming to the contrary. Then there must exists $\delta' > 0$ such that for each $n > 0$ there exist $x_n \in X$ with $x_n > n$, $h_n \in \mathcal{H}$ and $\psi_n \in \boldsymbol{u}$ satisfying $|h_n \odot \psi_n)(x_n) > \delta'$.

Let us define the sequence of functions on X by

$$S_n(x) = (h_n \odot \psi_n)_{x_n}(x) = (h_n \odot \psi_n)(xx_n),\ n \in \mathcal{N}.$$

We shall show that it is bounded and equicontinuous on X.

$$||S_n||_\infty = ||(h_n \odot \psi_n)x_n||_\infty = ||h_n \odot \psi_n||_\infty \leq ||h_n||_1||\psi_n||_\infty \leq ||h_o||_1\psi_n||_\infty \leq 1.$$

Therefore $\{S_n\}_n \in \mathcal{N}$ is bounded.

Let x, y $\in$ X, then

$$|S_n(x) - S_n(y)| = |(h_n \odot \psi_n)_{x_n}(x) - (h_n \odot \psi_n)_{x_n}(y)|$$

$$= |(h_n \odot \psi_n)_x(x_n) - (h_n \odot \psi_n)_y(x_n)|$$

$$= |(h_n \odot (\psi_n)_x(x_n) - (h_n \odot \psi_n)_y)(x_n)|$$

$$\leq ||h_n \odot(\psi_n)_x - (\psi_n)_y||_\infty$$

$$\leq ||h_n||_1||\psi_n)_x - (\psi_n)_y||_\infty \leq ||h_o||_1||(\psi_n)x - (\psi_n)_y||_\infty.$$

Since $x \to \psi_x$ is uniformly equicontinuous on X to $L^\infty(X)$, $\psi \in \boldsymbol{u}$ so given $\mathcal{E} > 0\ \exists\ \delta > 0$ such that for $|x - y| < \delta$ and $\psi \in \boldsymbol{u}$ we have, $||\psi_x - \psi_y||_\infty < \mathcal{E}$.

Therefore, $S_n(x) - S_n(y)| < \epsilon$ for $|x - y| < \delta$.

Hence, S_n is equicontinuous.

By Ascol's Theorem, there exists a subsequence (S_{n_k}) converging to a continuous function s pointwise.

For each x $\in$ X and t $\in$ X.

$$S_{n_k}\left(\frac{t}{x}\right) \to s\left(\frac{t}{x}\right)'\ k \to \infty$$

and therefore $S_{n_k}\left(\frac{t}{x}\right)g(t) = s\left(\frac{t}{x}\right)g(t)$ as $k \to \infty$.

Since $|S_{n_k}\left(\frac{t}{x}\right) g(t)| = \left|S_{n_k}\left(\frac{t}{x}\right)\right| |g(t)| \leq g(t)|$.

Thus by Lebesgue dominated convergence theorem,

$$\int_X S_{n_k}\left(\frac{t}{x}\right) g(t)d\mu(t) \to \int_X s\left(\frac{t}{x}\right) g(t)d\mu(t),\ k \to \infty$$

$$\equiv x(s \odot g)(x)$$

Now,

$$\begin{aligned}
\int_x S_{n_k}\left(\frac{t}{x}\right) g(t)d\mu(t) &= \int_X h_{n_k} \odot \psi_{n_k})\left(\frac{tx_{n_k}}{x}\right)g(t)d\mu t \\
&= \int_X \frac{x}{tx_{n_k}} \int_X h_{n_k}\left(\frac{xu}{tx_{n_k}}\right)\psi_{n_k}(u)g(t)\, d\mu(u)d\mu(t) \\
&= \int_X \frac{x}{tx_{n_k}} \int_X h_{n_k}(u)\psi_{n_k}\left(\frac{tx_{n_k}u}{x}\right)g(t)\left(\frac{tx_{n_k}}{x}\right)d\mu(u)\, d\mu(t) \\
&= \int_X \int_X h_{n_k}(u)\psi_{n_k}\left(\frac{tx_{n_k}u}{x}\right)g(t)d\mu(u)\, d\mu(t), \\
&= \int_x \frac{x}{x_{n_k}u} h_{n_k}(u)\left[\int_x g\left(\frac{tx}{x_{n_k}\,u}\right)\psi_{n_k}(t)d\mu(t)\right] d\mu(u) \\
&= \int_x h_{n_k}(u)(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right) d\mu(u) \\
&= \int_x F_{k,x}(u)d\mu(u)
\end{aligned}$$

Where $F_{k,x}(u) = (g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right) h_{n_k}(u)$.

Since we know, $(g \odot \psi)(t) \to 0$ as $t \to \infty$ uniformly for ψ in $\mathcal{u}$ we have for given $\mathcal{E} > 0$ there exists $\Delta : |(g \odot \psi)(t)| < \mathcal{E}$ for every $t \geq \Delta$ and ψ in $\mathcal{u}$.

Therefore,

$$\left|(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right)\right| \leq \mathcal{E} \text{ for } \left(\frac{x_{n_k}u}{x}\right) \geq \Delta$$

Thus for fixed x and u, $(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right) \to 0$ and $k \to \infty$, and therefore $F_{k,x}(u) \to 0$ as $k \to \infty$.

But $\|(g \odot \psi_{n_k})\|_\infty \le \|g\|_1\|\psi_{n_k}\|_\infty \le 1$ and therefore

$$|F_{k,x}(u)| = \left|(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right)h_{n_k}(u)\right| \le \left|(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right)\right| |h_{n_k}(u)| \le |h_{n_k}(u)| \le |h_o(u)|.$$

Thus again applying Lebesgue dominated convergence theorem $s \odot g(x) = 0$, since $g \in U$, $s = 0$.

But $0 = |s(1)| = \lim_{k\to\infty} s_{n_k}(1)| = \lim_{k\to\infty}| h_{n_k} \odot \psi_{n_k})(x_{n_k})|$.

Thus, we obtain a contradiction. Therefore

$$\frac{1}{x}\int_X h\left(\frac{t}{x}\right)\psi(t)d\mu(t) \to 0 \text{ as } x \to \infty$$

Uniformly for h in $\mathcal{H}$ and ψ in $\boldsymbol{u}$.

This completes the proof.

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