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# Cartesian product of $r$ hyperbolic Hermite manifolds

JAYA UPRETI AND SHANKAR LAL

**Abstract:** Cartesian Product of two manifolds has been defined and studied by Pandey [2]. In this paper we have taken Cartesian product of $r$ Hyperbolic Manifold, where $r$ is some finite integer and studied some properties of curvature and Ricci tensor of such a product manifold.

## 1. Introduction:

Let $M_1, M_2, \ldots, M_r$ be $r$ Hyperbolic Hermite structure manifolds each of class $C^\infty$ and of dimension $n_1, n_2, \ldots, n_r$ respectively. Suppose $(M_1)m_1, (M_2)m_2, \ldots, (M_r)m_r$ be their tangent spaces at $m_1 \in M_1, m_2 \in M_2, \ldots, m_r \in M_r$. Then the product space $(M_1)m_2 \times (M_2)m_2 \times, \ldots, \times (M_r)m_r$ contains vector fields of the form $(X_1, X_2, \ldots, X_r)$ where $X_1 \in (M_1)m_1, X_2 \in (M_2)m_2, \ldots, X_r \in (M_r)m_r$ [2]. Vector addition and scalar multiplication on above product space are defined as follows:

$$(1.1) \qquad (X_1, X_2, \ldots, X_r) + (Y_1, Y_2, \ldots, Y_r) = (X_1 + Y_2, X_1 + Y_2, \ldots, X_r + Y_r)$$

$$(1.2) \qquad \lambda(X_1, X_2, \ldots, X_r) = (\lambda X_1, \lambda X_2, \ldots, \lambda X_r)$$

Where $X_i, Y_i \in (M_i)m_i$, $i = 1, 2, \ldots, r$ and $\lambda$ is a scalar.

Under these conditions the product space $(M_1)m_1 \times (M_2)m_2 \times \ldots \times (M_r)m_r$ forms a vector space.

Define a linear transformation $F$ on the product space.

$$(1.3) \qquad F(X_1, X_2, \ldots, X_r) = (F_1X_1, F_2X_2, \ldots, F_rX_r)$$

Where $F_1, F_2, \ldots, F_r$ are linear transformations on $(M_1)m_1, (M_2)m_2, \ldots, (M_r)m_r$ respectively

If $f_1, f_2, \ldots, f_r$ be $C^\infty$ functions over the spaces $(M_1)m_1, (M_2)m_2, \ldots, (M_r)m_r$ respectively, we define the $C^\infty$ function $(f_1, f_2, \ldots, f_r)$ on the product as

$$(1.4) \qquad (X_1, X_2, \ldots, X_r)(f_1, f_2, \ldots, f_r) = (X_1f_1, X_2f_2, \ldots, X_rf_r)$$

Let If $D_1, D_2, \ldots, D_r$ be the connection on the Manifolds $M_1, M_2, \ldots, M_r$ respectively. We define the operator D on the product space as

$$(1.5) \qquad D_{(X_1X_2,\ldots,X_r)}(Y_1, Y_2, \ldots, Y_r) = D_{1x_1}, D_{2x_2}, \ldots, D_{rx_r})$$

Then $D$ satisfies all four properties of a connection and thus it is a connection on the product manifold.

## 2. Some Results

**Theorem 2.1.** *The product manifold $M_1 \times M_2 \times \ldots \times M_r$ admits an almost Hyperbolic Hermite structure If and only if the manifolds $M_1, M_2, \ldots, M_r$ are almost Hyperbolic Hermite structure manifolds.*

**Proof:** Suppose $M_1, M_2, \ldots, M_r$ are Hyperbolic Hermite structure manifolds. Thus there exist tensor fields $F_1, F_2, \ldots, F_r$ each of type (1.1) on $M_1, M_2, \ldots, M_r$ respectively satisfying

$$F_i^2(X_i) = X_i \quad i = 1, 2, \ldots, r. \tag{2.1}$$

In view of equation (1.3) it follows that there exists a linear transformation $F$ on $M_1 \times M_2 \ldots \times M_r$ satisfying

$$F^2(X_1, X_2, \ldots, X_r) = (F_1^2 X_1 F_2^2, X_2, \ldots, F_r^2 X_r) = (X_1, X_2, \ldots, X_r) \tag{2.2}$$

Let us define a Riemannian metric $g$ on the product manifold $M_1 \times M_2 \times \ldots \times M_r$ as

$$g((X_1, X_2, \ldots, X_r), (Y_1, Y_2, \ldots, Y_r)) = g_1(X_1, Y_1) + g_2(X_2, Y_2) + \ldots + g_r(X_r, Y_r) \tag{2.3}$$

Where

$$\begin{aligned} g((FX_1, FX_2, \ldots, FX_r), (FY_1, FY_2, \ldots, FY_r)) = &- g_1(X_1, Y_1) \\ &- g_2(X_2, Y_2) - \ldots - g_r(X_r, Y_r) - \{\eta(X_1)\eta(Y_1) + \\ &+ \eta(X_2)\eta(Y_2) + \ldots + \eta(X_r)\eta(Y_r)\} \end{aligned} \tag{2.4}$$

and $g_1, g_2, \ldots, g_r$ are Riemannian metrics over the manifold $M_1 \times M_2 \times \ldots \times M_r$ respectively. Thus the product space admits an almost Hyperbolic Hermite structure [1].

If $\xi_1, \xi_2, \ldots, \xi_r$ be the vector fields and $\eta_1, \eta_2, \ldots, \eta_r$ be 1-form on the almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$ respectively then a vector field $\xi$ and a 1-form $\eta$ on the product manifold is defined as

$$\eta(X)\xi = (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \ldots, \eta_r(X_r)\xi_r) \tag{2.5}$$

We now prove the following results.

**Theorem 2.2.** *The Hyperbolic product manifold $M_1 \times M_2 \times \ldots \times M_r$ admits Hyperbolic contact structure if and only if the manifold $M_1, M_2, \ldots, M_r$ possess the same structure.*

**Proof:** Let $M_1, M_2, \ldots, M_r$ are Hyperbolic almost contact manifolds. Thus there exists tensor fields $F_i$, of type (1,1), vector fields $\xi_i$ and 1-forms. $\eta_i$, $i = 1, 2, \ldots, r$

$$F_i^2(X_i) = X_i + \eta_i(X_i)\xi_i \tag{2.6}$$

For product manifold $M_1 \times M_2 \times \ldots \times M_r$

$$F^2(X_1, X_2, \ldots, X_r) = (P_1^2 X_1, X_2, F_2^2 X_2, \ldots, F^2, X_r)$$

which is view of (2.5) and (2.6) takes the form

$$F^2(X_1,X_2,\ldots,X_r)=(X_1,X_2,\ldots,X_r)+(\eta_1(X_1)\xi_1,\eta_2(X_2)\xi_2,\ldots,\eta_r(X_r)\xi_r)$$

or $$F^2(X)=X+\eta(X)\xi$$

Hence product manifold admits a Hyperbolic almost contact metric structure [3].

**Theorem 2.3.** *The Hyperbolic product manifold $M_1\times M_2\times\ldots\times M_r$ admits Hyperbolic Kahler structure if and only if the manifolds $M_1, M_2,\ldots,M_r$ are Kahler manifolds.*

**Proof:** Suppose $M_1, M_2,\ldots,M_r$ are Kahaler manifolds. Then

(2.7) $$(D_{1x_1}F_1)(Y_1)=(D_{2x_2}F_2)(Y_1)=\ldots=(D_{rx_r}F_r)(Y_r)=0$$

As $D$ is a connection on the product manifold. Hence

(2.8) $$D_{(x_1x_2,\ldots,x_r)}F)(Y_1,Y_2,\ldots,Y_r)=D_{(x_1x_2,\ldots,x_r)}\{F(Y_1,Y_2,\ldots,Y_r)-$$
$$-F\{D_{(x_1x_2,\ldots,x_r)}(Y_1,Y_2,\ldots,Y_r)\}$$

Which in view of equation (1.3), equation (1.5) takes the form

$$D_{(x_1x_2,\ldots,x_r)}F)(Y_1,Y_2,\ldots,Y_r)=D_{(x_1x_2,\ldots,x_r)}\{F_1Y_1,F_2Y_2,\ldots,F_rY_r)-$$
$$-F(D_{1x_1}Y_1,D_{2x_2}Y_2,\ldots,D_{rx_r}Y_r)$$
$$=(D_{1x_1}F_1Y_1,D_{2x_2}F_2Y_2,\ldots,D_{rx_r}F_rY_r)-(F_1D_{1x_1}Y_1,F_2D_{2x_2}Y_2,\ldots,F_rD_{rx_r}Y_r)$$
$$=((D_{1x_1}F_1),(D_{2x_2}F_2)(Y_2),\ldots,(D_{rx_r}F_r)(Y_r))$$
$$=0.$$

Thus the product manifold is Hyperbolic Kahler structure manifold.

**Theorem 2.4.** *The product manifold $M_1\times M_2\times\ldots\times M_r$ of a Hyperbolic almost contact metric structure manifolds $M_1, M_2,\ldots,M_r$ is almost Tachibana if and only if the manifolds $M_1, M_2,\ldots,M_r$ are separately Tachibana manifolds.*

**Proof:** Let an almost Hyperbolic Hermite structure manifolds $M_1, M_2,\ldots,M_r$ are almost Tachibana manifolds. Then

(2.9) $$(D_{iX_i}F_i)(Y_i)+(D_{iX_i}F_i)(Y_i)=0, i=1,2,\ldots,r$$

The result follows in view of the previous theorem (2.3)

## 3. Curvature and Ricci tensor

Let $X=(X_1,X_2,\ldots,X_r)$ and $Y=(Y_1,Y_2,\ldots,Y_r)$ be $C^\infty$ vector fields on the product manifold $M_1\times M_2\times\ldots\times M_r$ and $f=(f_1,f_2,\ldots,f_r)$ be a $C^\infty$ function. Then

$$[(X_1,X_2,\ldots,X_r),(Y_1,Y_2,\ldots,Y_r)](f_1,f_2,\ldots,f_r)$$
$$=(X_1,X_2,\ldots,X_r)\{(Y_1,Y_2,\ldots,Y_r)$$

(3.1) $$(f_1,f_2,\ldots,f_r)\}-(Y_1,Y_2,\ldots,Y_r)\{(X_1,X_2,\ldots,X_r),(f_1,f_2,\ldots,f_r)\}.$$
$$=([X_1,Y_1]f_1,[X_1,Y_2]f_2,\ldots,[X_r,Y_r]f_r).$$

Suppose $K_i[X_i, Y_i, Z_i)$, $i = 1,2, \ldots, r$ be the curvature tensors of the almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$ respectively. If $K(X, Y, Z)$ be the Curvature tensor of the product manifold $M_1 \times M_2 \times \ldots \times M_r$.
Then we have

$$K(X, Y, Z) = [K_1(X_1, Y_1, Z_1], K_2(X_2, Y_2, Z_2), \ldots, K_r(X_r, Y_r, Z_r)]. \tag{3.2}$$

Let $W = (W_1, W_2, \ldots, W_r)$ be a vector field on the product manifold. Then

$$K'(X, Y, Z, W) = g(K(X, Y, Z) W) \tag{3.3}$$

$$K'(X, Y, Z, W) = K'_1(X_1, Y_1, Z_1, W_1) + K'_2(X_2, Y_2, Z_2, W_2) + \ldots + K'_r(X_r, Y_r, Z_r, W_r) \tag{3.4}$$

Then we have

**Theorem 3.1.** *The product manifold $M_1 \times M_2 \times \ldots \times M_r$ is of constant curvature if and only if almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$ are separately of constant curvature.*

**Theorem 3.2.** *The Ricci tensor of the product manifold $M_1 \times M_2 \times \ldots \times M_r$ is the sum of the Ricci tensor of the almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$.*

**Theorem 3.3.** *The product manifold $M_1 \times M_2 \times \ldots \times M_r$ is an Einstein space if and only if almost Hyperbolic Hermite manifolds $M_1, M_2, \ldots, M_r$ are separately Einstein spaces.*

**Proof:** Let the product manifold $M_1 \times M_2 \times \ldots \times M_r$ is an Einstein space. Thus

$$\text{Ric}(X, Y) = Cg(X, Y) \tag{3.5}$$

Where $C = \frac{K}{n}$, $K$ being the scalar curvature and $n$ being the dimension of the product manifold. Then

$$\text{Ric}(X_i, Y_i) = Cg_i(X_i Y_i), \quad i = 1,2, \ldots, r.$$

Therefore the manifolds $M_1, M_2, \ldots, M_r$ are also Einstein spaces.

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**JAYA UPRETI AND SHANKAR LAL**
Department of Mathematics
Kumaun University,
S.S.J. Campus,
Almora, Uttaranchal, India.