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Certain transformation formulae of hypergeometric type

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Abstract: The object of this paper is to be established certain transformation formulae involving hypergeometric series by making use of Bailey's transformation.

Key words and phrases: Hypergeometric series, Kampe de Feriet hypergeometric function, summation formulae and transformation formulae.

1. Introduction

An infinite series of the form $\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{(1)_n}$ is known as ordinary hypergeometric series. It is denoted as ${}_2F_1[a,b\,;c\,;z]$, where a, b and c are real or complex parameters and z is an argument with $|z| < 1$. The hypergeometric series (GHS) have been the topic of a significant study by W.N. Bailey, R.P. Agarwal, L. J. Slater and more recently G. Gasper and M. Rahman in its general form also.

A generalized hypergeometric series is defined as

$$(1.1)\qquad {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r\,; \\ b_1, b_2, \ldots, b_s, z \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n, \ldots, (a_r)_n\, z^n}{(b_1)_n (b_2)_n, \ldots, (b_s)_n (1)_n},$$

where

$$(a)_n = \begin{cases} a(a+1), \ldots, (a+n-1), & n = 1, 2, \ldots \\ 0, & n = 0. \end{cases}$$

The series (1.1) converges for all z when $r \le s$, at least when none of denominator parameters are zero or negative integer. It converges for $|z| < 1$ when $r = s + 1$ and only converges for $z = 0$ when $r > p + 1$ unless it reduces to a polynomial.

A kampe de Feriet hypergeometric function of two variables is defined as

$$(1.2)\qquad F^{\lambda;\mu;\mu'}_{\rho;\upsilon;\upsilon'}\left[\begin{matrix} (a_\lambda) : (b_\mu), (B'_{\mu'})\,; x, y \\ (c_\rho), (d_\upsilon), (d'_{\upsilon'}) \end{matrix}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[(a_\lambda)]_{m+n}[(b_\mu)]_m[(b'_{\mu'})]_n\, x^m y^n}{[(c_\rho)]_{m+n}[(d_\upsilon)]_m[(d'_{\upsilon'})]_n\,(1)_m\,(1)_n},$$

where $(a_\lambda)_m$ stand for the sequence of parameters $a_1, a_2, \ldots, a_\lambda$ and $|x|<1$, $|y|<1$, and $\lambda+\mu'+\mu'<\rho+\upsilon+\upsilon'+1$ for convergence.

2. In order to establish certain transformation and summation formulae for generalized hypergeomtric series, we shall make use of the following Bailey's transformation:

If

(2.1) $$\beta_n = \sum_{r=0}^{n} \alpha_r\, U_{n-r}\, V_{n+r} \quad \text{and}$$

(2.2) $\gamma_n = \sum_{r=0}^{n} \delta_{r+n}\, u_r\, v_{r+2n}$, where α_r, δ_r, u_r and v_r are the functions of r only such that the series γ_n exists then under suitable convergence conditions:

(2.3) $$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$$

Now, we shall be in need of the following known relations due to Slater [4]:

(2.4) $${}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d, 1+2a-b-c-d+n, -n; 1 \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, b+c+d, 1+a+n \end{matrix}\right]$$
$$= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}$$

(2.5) $${}_3F_2\left[\begin{matrix} b, -n; 1 \\ 1+a-b, 1+a+n \end{matrix}\right] = \frac{(1+a)_n\left(1+\frac{a}{2}-b\right)_n}{\left(1+\frac{a}{2}\right)(1+a-b)_n}$$

(2.6) $${}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}a, b, -n; 1 \\ \frac{1}{2}a, 1+a-b, 1+a+n \end{matrix}\right] = \frac{(1+a)_n}{(1+a-b)_n}$$

(2.7) $${}_3F_2\left[\begin{matrix} a, b, -n; 1 \\ 1+a-b, a+2b-n \end{matrix}\right] = \frac{(a-2b)_n\left(1+\frac{a}{2}-b\right)_n(-b)_n}{(1+a-b)_n\left(\frac{a}{2}-b\right)_n\left(\frac{a}{2}-b\right)_n(-2b)_n}$$

(2.8) $${}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, -n; 1 \\ \frac{a}{2}, 1+a-b, 1+2b-n \end{matrix}\right] = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}$$

(2.9) $${}_2F_1\left[\begin{matrix} a,b\,;1 \\ 1+a-b \end{matrix}\right] = \frac{(1+a)_n(1+b)_n}{(1+a+b)_n(1)_n}$$

(2.10) $${}_2F_1\left[\begin{matrix} a,b\,;1 \\ c \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

3. Main Results.

(3.1) If we take

$$u_n = \frac{(1+a-b-c-d)_n}{(1)_n},$$

$$v_n = \frac{(1+2a-b-c-d)_n}{(1+a)_n},$$

and $$\alpha_n = \frac{(a)_n\left(1+\frac{a}{2}\right)_n(b)_n(c)_n(d)_n}{(1)_n\left(\frac{a}{2}\right)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}$$

in (2.1), then we have

$$\beta_n = \sum_{r=0}^{n}\frac{(a)_r\left(1+\frac{a}{2}\right)_r(b)_r(c)_r(d)_r}{(1)_r\left(\frac{a}{2}\right)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r} \times\frac{(1+a-b-d)_{n-r}(1+2a-b-d)_{n+r}}{(1)_{n-r}(1+a)_{n+r}}$$

$$= \frac{(1+a-b-c-d)_n(1+2a-b-c-d)_n}{(1)_n(1+a)_n}$$

$$\times\sum_{r=0}^{n}\frac{(a)_r\left(1+\frac{a}{2}\right)_r(b)_r(c)_r(d)_r(1+2a-b-c-d+n)_r}{(1)_r\left(\frac{a}{2}\right)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r} \times\frac{(-n)_r}{(b+c+d-n)_r(1+a+n)_r}$$

$$= \frac{(1+a-b-c-d)_n(1+2a-b-c-d)_n}{(1)_n(1+a)_n}$$

(2.4) $$\times{}_7F_6\left[\begin{matrix} a,1+\frac{a}{2},b,c,d,1+2a-b-c+n,-n;1 \\ \frac{a}{b},1+a-b,1+a-c,1+a-d,b+c+d-n,1+a+n; \end{matrix}\right]$$

We shall now make use of (2.4) to obtain.

$$\beta_n = \frac{(1+2a-b-c-d)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}$$

Putting the values of u_n and v_n in (2.2) and taking $\delta_n = 1$, we have

$$\gamma_n = \sum_{r=0}^{\infty} 1, \frac{(1+a-b-c-d)_r(1+2a-b-c-d)_{r+2n}}{(1)_r(1+a)_{r+2n}}$$

$$= \frac{(1+2a-b-c-d)_{2n}}{(1+a)_{2n}} \times \sum_{r=0}^{\infty} \frac{(1+a+b-c-d)_r(1+2a-b-c-d+2n)_r}{(1)_r(1+a+2n)_r}$$

$$= \frac{(1+2a-b-c-d)}{(1+a)_{2n}} {}_2F_1\left[\begin{matrix} 1+a-b-cd, 1+2a-b-c-d+2n; 1 \\ 1+a+2n \end{matrix}\right]$$

Also, making use of (2.10), we get

$$\gamma_n = (1+2a-b-c-d)_{2n} \frac{\Gamma(1+a)\Gamma(2b+2c+2d-2a-1)}{\Gamma(b+c+d)\Gamma(b+c+d-a)},$$

provided $\mathrm{Re}(2b+2c+2d-2a-1) > 0$.

Putting the values of $\beta_n, \gamma_n, \alpha_n$ and δ_n in (2.3), we get

$$\sum_{n=0}^{\infty} \frac{(a)_n\left(1+\frac{a}{2}\right)_n(b)_n(c)_n(d)_n}{(1)_n\left(\frac{a}{2}\right)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}$$

$$\times \frac{(1+2a-b-c-d)_{2n}\Gamma(1+a)\Gamma(2b+2c+2d-2a-1)}{(b+c+d)_{2n}\Gamma(b+c+d)\Gamma(b+c+d-a)}$$

$$= \sum_{n=0}^{\infty} \frac{(1+2a-b-c-d)_n(1+a-b-c)_n(1+a-b-d)_n(1+a=c=d)_n}{(1)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}$$

$$\times \frac{\Gamma(1+a)\Gamma(2b+2c+2d-2a-1)}{\Gamma(b+c+d)\Gamma(b+c+d-a)}$$

$$\times {}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d-\frac{b}{2}-\frac{c}{2}-\frac{d}{2}, 1+a-\frac{b}{2}-\frac{c}{2}-\frac{d}{2}; 1 \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, \frac{b}{2}+\frac{c}{2}+\frac{d}{2}, \frac{1}{2}+\frac{b}{2}+\frac{c}{2}+\frac{d}{2}; \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} 1+2a \quad b-c-d, 1+a-b-c, 1+a-b-d, 1+a-c-d; 1 \\ 1+a-b, 1+a-c, 1+a-d \end{matrix}\right]$$

provided $\mathrm{Re}(2b+2c+2d-2a) > 0$

(3.2) If we take

$$u_n = \frac{1}{(1)_n}, \; v_n = \frac{1}{(1+a)_n} \text{ and } \alpha_n = \frac{(a)_n (b)_n (-1)^n}{(1)_n (1+a-b)_n} \text{ in (2.1), then we have}$$

$$\beta_n = \sum_{r=0}^{n} \frac{(a)_r\,(b)_r(-1)^r}{(1)_r\,(1+a-b)_r(1)_{n-r}(1+a)_{n+r}}$$

$$= \frac{1}{(1)_n\,(1+a)_n} \sum_{r=0}^{n} \frac{(a)_r\,(b)_r(-n)_r}{(1)_r\,(1+a-b)_r\,(1+a+n)_r}$$

$$= \frac{1}{(1)_n\,(1+a)_n}\; {}_3F_2\left[\begin{matrix} a,b,-n\,; \\ 1+a-b,1+a-n\,; \end{matrix}\right],$$

Now, making use of (2.5), we get

$$\beta_n = \frac{\left(1+\frac{a}{2}-b\right)_n}{n!\left(1+\frac{a}{2}\right)_n (1+a-b)_n}$$

Again, putting the values of u_n, v_n in (2.2) and taking $\delta_n = (\alpha)_n(\beta_n)$, we have

$$\gamma_n = \sum_{r=0}^{\infty} (\alpha)_{r+n}\,(\beta)_{r+n}\,\frac{1}{(1)_r} \times \frac{1}{(1+a)_{r+2n}}$$

$$= \frac{(\alpha)_n\,(\beta)_n}{(1+a)_{2n}} \sum_{r=0}^{\infty} \frac{(a+n)_r\,(\beta+n)_r}{(1)_r\,(1+a+2n)_r}$$

$$= \frac{(\alpha)_n\,(\beta)_n}{(1+a)_{2n}}\; {}_2F_1\left[\begin{matrix} \alpha+n\,,\beta+n\,;1 \\ 1+a+2n\,; \end{matrix}\right]$$

Finally, using (2.10), we have

$$\gamma_n = \frac{\Gamma(1+a)\,\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\,\Gamma(1+a-\beta)} \cdot \frac{(\alpha)_n\,(\beta)_n}{(1+a-\alpha)_n(1+a-\beta)_n},$$

Provided $\mathrm{Re}(1+a-\alpha-\beta) > 0$

Putting the values of $\alpha_n, \beta_n, \gamma_n$ and δ_n in (2.3), we get

$$\frac{\Gamma(1+\alpha)\,\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\,\Gamma(1+a-\beta)} \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n(\alpha)_n\,(\beta)_n(-1)_n}{(1)_n(1+a-b)_n(1+a-\alpha)_n(1+a-\beta)_n}$$

$$= \sum_{n=0}^{\infty} \frac{\left(1+\frac{a}{2}-b\right)_n (\alpha)_n\,(\beta)_n}{(1)_n\left(1+\frac{a}{2}\right)_n (1+a-b)_n} \times {}_4F_3\left[\begin{matrix} a,b,\alpha,\beta\,;1 \\ 1+a-b,1+a-\alpha\,,1+\alpha-\beta \end{matrix}\right]$$

$$= \frac{\Gamma(1+a-\alpha)\,\Gamma(1+a-\beta)}{\Gamma(1+a)\,\Gamma(1+a-\alpha-\beta)} \times {}_3F_2\left[\begin{matrix}\alpha, \beta, 1+\frac{a}{2}, -b\,;1\\ 1+\frac{a}{2}, 1+a-b\end{matrix}\right]$$

provided $\mathrm{Re}(1+a-\alpha-\beta)>0$

Similarly, making use of (2.6) – (2.9), we can establish the other transformation formulae with the help of (2.10).

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