The Nepali Math. Sc. Report
Vol. 26, No. 1 & 2, 2006.

Common fixed point theorems for four maps in D-metric space using certain continuity conditions

K.P.R. RAO[1], K. JHA[2] AND P. RANGA SWAMY[3]

Abstract: In this paper we prove two common fixed point theorems; for four self maps on Dhage metric space using certain orbitally lower semi continuity condition on four maps; which are some probable modifications of theorems of Dhage and Dhage et al.

AMS 2000 Subject Classification: 47 H 10; 54 H 25

Key Words and Phrases: *D*-metric spaces, fixed point, coincidently commuting maps, orbitally lower semi-continuous maps.

Dhage [1] introduced the concept of *D*-metric space as follows.

Definition: A non empty set X, together with a function $D : X \times X \times X \to [\,0, \infty\,)$ is called a *D*-metric space with *D*-metric satisfies the following properties

(i) $D(x,y,z) = 0$ iff $x = y = z$

(ii) $D(x, y, z) = D(p\{x, y, z\})$ where p is a permutation function of x, y, z

(iii) $D(x,y,z) \le D(a,y,z) + D(x,a,z) + D(x,y,a)\ \forall x,y,z,a \in X.$

Definitions [1]: A sequence $\{x_n\} \subset X$ is said to be convergent to a point $x \in X$ if $\underset{m,n\to\infty}{\mathrm{Lt}}\ D(x_m, x_n, x) = 0.$

A sequence $\{x_n\} \subset X$ is called *D*-Cauchy if $\underset{m,n,p\to\infty}{\mathrm{Lt}}\ D(x_m, x_n, x_p) = 0$

A complete *D*-metric space is one in which every *D*-Cauchy sequence converges to a point of X. A subset E of a *D*-metric space X is called bounded if there exists a constant $k > 0$ such that $D(x, y, z) \le k\ \forall x, y, z \in$ E. A mapping $f : X \to X$ is continuous if and only if, for any sequence $\{x_n\} \subseteq X$, $x_n \to x$ implies $fx_n \to fx$.

Dhage [1] also claimed that *D*-metric is continuous in all its three variables.

Dhage [2] proved the following

Lemma 1 (Lemma 2.2 [2]) : Let $\{x_n\} \subseteq X$ be bounded with *D*-bound k satisfying $D(x_n, x_{n-1}, x_m) \le \phi^n(k)\ \forall m > n \in \mathrm{N}$ where $\phi : \mathrm{R}^+ \to \mathrm{R}^+$ satisfies $\sum_{n=1} \phi^n(t) < \infty$ for each $t \in \mathrm{R}^+$. Then $\{x_n\}$ is *D*-Cauchy.

Theorem 2 (Theorem 2.1 of [2]): *Let S and T be self maps on a D-metric space X, X be (S,T) - orbitally complete and (S,T) - orbitally bounded D-metric space.*

Suppose that

$D(Sx,Ty,z) \le \phi(\text{Max}\{D(x,y,z),\ D(x,Sx,z),\ D(y,Ty,z),\ \beta D(x,Ty,z),$
$\beta D(y,Sx,z)\})$ for all $x,y \in X$ and $z \in \overline{O(S,T;x) \cup O(T,S;y)}$ where $0 \le \beta \le 1/3$ and
$\phi: R^+ \to R^+$ is continuous, non decreasing, $\phi(t) < t$ for $t > 0$ and $\sum_{n=1} \phi^n(t) < \infty\ \forall t \in R^+$.

Then S and T have a unique common fixed point.

We observed that in Theorem 2, the author used the continuity of D-metric in one variable in proving the existence of fixed point of T. But Naidu et.al. [4] observed that there are D-metrices which are not continuous even in a single variable.
(See Example 3 of [4]). Hence the validity of Theorem 2 is doubtful.

Dhage et.al. [3] proved

Theorem 3 (Theorem 2.4 of [3]): *Let A,B,S T be four self maps of a D-metric space X satisfying*

(3.1) $A(X) \subseteq T(X),\ \ B(X) \subseteq T(S)$

(3.2) $D(Ax,By,z) \le \lambda\, \text{Max}\{D(Sx,Ty,z),\ D(Sx,Ax,z),\ D(Ty,By,z)\}\ \forall x,y,z \in X$ where $0 \le \lambda < 1$

(3.3) $\overline{O_{A,B}(S,T,x)}$ is complete for each $x \in X$

(3.4) the pairs $\{A,S\}$ and $\{B,T\}$ are limit coincidentally commuting

(3.5) any one of A,B,S,T is continuous

Then A, B, S and T have a unique common fixed point.

In this theorem also Dhage et. al [3] used the continuity of D-metric in two variables. Hence the validity of this theorem is also doubtful. Even if this theorem is valid, the inequality (3.2) forces the space X to be a singleton set. Hence this theorem is insignificant.

Now we give some modifications of Theorems 2 and 3 without using the continuity of D-metric.

We first give the following definitions

Let A,B,S and T be four self maps on a D-metric space X.

Let $G(x) = \min\{D(Ax,Ax,Sx), D(Ax,Sx,Sx), D(Bx,Bx,Tx), D(Bx,Tx,Tx)\}$,

$G^*(x) = \max\{D(Ax,Ax,Sx), D(Ax,Sx,Sx), D(Bx,Bx,Tx), D(Bx,Tx,Tx)\}$,

$H_1(x) = \max\{D(Ax,Ax,Sx), D(Ax,Sx,Sx)\}$,

$H_2(x) = \max\{D(Bx,Bx,Tx), D(Bx,Tx,Tx)\}$.

We say that A, B, S and T are (G,H_1,H_2)- orbitally lower semi continuous at $u \in X$ if

$$G(u) \le \underline{\lim_{n\to\infty}}\ \max\{H_1(x_{2n}), H_2(x_{2n+1})\}$$

whenever there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_0 = Sx_0,\ y_{2n+1} = Ax_{2n} = Tx_{2n+1},$$
$$y_{2n+2} = Bx_{2n+1} = Sx_{2n+2},\ n = 0,1,2,\ldots$$

and $\{y_n\}$ converges to u

We prove the following two lemmas

Lemma 4: Let A, B, S and T be four self maps on a D-metric space (X, D) satisfying

(4.1) $D(Ax, By, z) \le \phi(\text{Max}\{D(Sx, Ty, z), D(Sx, Ax, z), D(Ty, By, z), D(Sx, By, z), D(Ty, Ax, z)\}\ \forall x, y \in X$ and $z = Az_1$ or Bz_2 for some $z_1, z_2 \in X$ where $\phi: R^+ \to R^+$ is a mapping such that $\phi(t) < t\ \forall t > 0$.

Further suppose that

(4.2) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$

(4.3) there exists $u \in X$ such that $Au = Su$ or $Bu = Tu$

(4.4) the pairs $\{A, S\}$ and $\{B, T\}$ are coincidentally commuting at u

Then A, B, S and T have a unique common fixed point.

Proof: Suppose $Au = Su$

Since $A(X) \subseteq T(X)$ there exists $v \in X$ such that $Au = Tv$.
Suppose $Au \ne Bv$

$$D(Au, Bv, Au) \le \phi(\text{Max}\{D(Su, Tv, Au), D(Su, Au, Au), D(Tv, Bv, Au), D(Su, Bv, Au), D(Tv, Au, Au)\}$$

$$= \phi(D(Au, Bv, Au)) < D(Au, Bv, Au).$$ It is a contradiction.

Hence $Au = Bv$

Thus $Au = Su = Bv = Tv$ ... ... (I)

Since the pair (A, S) is coincidentally commuting we have

$$A(Au) = A(Su) = S(Au) = S(Su) \quad \text{... ... (II)}$$

Since the pair (B, T) is coincidentally commuting we have

$$B(Bv) = B(Tv) = T(Bv) = T(Tv) \quad \text{... ... (III)}$$

$$D(Au, Au, A^2u) = D(Av, Bv, A^2u) \text{ from (I)}$$

$$\le \phi(\text{Max}\{D(Su, Tv, A^2u), D(Su, Au, A^2u), D(Tv, Bv. A^2u), D(Su, Bv, A^2u), D(Tv, Au, A^2u\} \text{ from (4.1)}$$

$$= \phi(D(Au, Au, A^2u)) \text{ from (I)}$$

$\therefore A^2u = Au$

Hence $S(Au) = A^2u = Au$ from (II)

Thus Au is a common fixed point of A and S.

$$D(Bv, Bv, B^2v) = D(Au, Bv, B^2v) \text{ from (I)}$$

$$\le \phi(\text{Max}\{D(Su, Tv, B^2v), D(Su, Au, B^2v), D(Tv, Bv, B^2v), D(Su, Bv, B^2v), D(Tv, Au, B^2v)\} \text{ from (4.1)}$$

$$= \phi(D(Bv, Bv, B^2v)) \text{ from (I)}$$

$\therefore B^2v = Bv$

Hence $T(Bv) = B^2 v = Bv$ from (III)

Thus Bv is a common fixed point of B and T.

Since $Au = Bv$ it follows that Au is a common fixed point of A, B, S, and T.

Similarly if $Bu = Tu$ then Bu is a common fixed point of A, B, S, and T.

Uniqueness of common fixed point follows easily by applying (4.1) two times.

Lemma 5: Let A, B, S, and T be four self maps on a D-metric space X satisfying

(5.1) $D(Ax, By, z) \le \phi(\text{Max}\{D(Sx, Ty, z), D(Sx, Ax, z), D(Ty, By, z), D(Sx, By, z), D(Ty, Ax, z)\}\ \forall x, y \in X$ and $z = x$ or y or Az_1 or Bz_2 for some $z_1, z_2 \in X$.

where $\phi : R^+ \to R^+$ is a function such that $\phi(t) < t \quad \forall t > 0$.

(5.2) Suppose $Au = Su$ and $Tu = Bu$ for some $u \in X$.

Then u is the common fixed point of A, B, S and T.

Proof: Write $w_1 = Au = Su, w_2 = Tu = Bu$

Suppose $w_1 \ne w_2$

$$\begin{aligned} D(w_1, w_2, w_1) &= D(Au, Bu, w_1) \\ &\le \phi(\text{Max}\{D(Su, Tu, w_1), D(Su, Au, w_1), D(Tu, Bu, w_1) \\ &\qquad D(Su, Bu, w_1), D(Tu, Au, w_1), \\ &= \phi(\text{Max}\{D(w_1, w_2, w_1), D(w_2, w_2, w_1)\}) \end{aligned}$$

Also

$$\begin{aligned} D(w_1, w_2, w_2) &= D(Au, Bu, w_2) \\ &\le \phi(\text{Max}\{D(Su, Tu, w_2), D(Su, Au, w_2), D(Tu, Bu, w_2), \\ &\qquad D(Su, Bu, w_2), D(Tu, Au, w_2)\} \\ &= \phi(\text{Max}\{D(w_1, w_2, w_2), D(w_1, w_1, w_2)\}) \end{aligned}$$

$$\therefore\ \text{Max}\{D(w_1, w_2, w_2), D(w_1, w_2, w_2)\} \le \phi(\text{Max}\{D(w_1, w_2, w_2), D(w_1, w_1, w_2)\})$$

Hence $w_1 = w_2$.

Thus $Au = Su = Bu = Tu$

Now

$$\begin{aligned} D(Au, Au, u) &= D(Au, Bu, u) \\ &\le \phi(\text{Max}\{D(Su, Tu, u), D(Su, Au, u), D(Tu, Bu, u), \\ &\qquad D(Su, Bu, u), D(Tu, Au, u)\} \\ &= \phi(D(Au, Au, u)) \end{aligned}$$

$\therefore Au = u$

Thus $Au = Bu = Su = Tu = u$

Hence u is a common fixed point of A, B, S and T.

Uniqueness of common fixed point follows easily from (5.1) using two times.

Now we give our main theorems.

Theorem 6: *Let A, B, S and T be four self maps on a D-metric space X such that*

(6.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(6.2) $D(Ax,By,z) \leq \phi(\text{Max}\{D(Sx,Ty,z),D(Sx,Ax,z),D(Ty,By,z),$
$\beta D(Sx,By,z), \beta D(Ty,Ax,z)\})$

$\forall x,y \in X, z = Az_1$ or Bz_2 for some $z_1,z_2 \in X$ where $0 \leq \beta \leq 1/3$ and $\phi: R^+ \to R^+$ is non decreasing, $\phi(t) < t$ for $t > 0$ and $\sum_{n=1} \phi^n(t) < \infty \ \forall \ t \in R^+$

(6.3) Assume that for some $x_0 \in X$, there exist sequence $\{x_n\}$ and $\{y_n\}$ such that $y_0 = Sx_0, y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}, n = 0,1,2\ldots$ and the set $D(y_a,y_b,y_m)/ 0 \leq a \leq 1, 1 \leq b \leq 2, m \geq 2\}$ is bounded by k.

Then the sequence $\{y_n\}$ is D-Cauchy.

Further assume that

(6.4) $\{y_n\}$ converges to some $u \in X$

(6.5) the pairs $\{A,S\}$ and $\{B,T\}$ are coincidentally commuting at u

(6.6) A,B,S,T are (G,H_1,H_2)- orbitally lower semi continuous at u

Then A, B, S and T have a unique common fixed point.

Proof : For any $m \geq 2$,

$$\begin{aligned}D(y_1,y_2,y_m) &= D(Ax_0,Bx_1,y_m)\\ &\leq \phi(\text{Max}\{D(Sx_0,Tx_1,y_m),D(Sx_0,Ax_0,y_m),D(Tx_1,Bx_1,y_m),\\ &\qquad \beta D(Sx_0,Bx_1,y_m),\beta D(Tx_1,Ax_0,y_m)\})\\ &= \phi(\text{Max}\{D(y_0,y_1,y_m),D(y_0,y_1,y_m),D(y_1,y_2,y_m),\\ &\qquad \beta D(y_0,y_2,y_m),\beta D(y_1,y_1,y_m)\})\\ &\leq \phi(k)\ldots\text{(i) from (6.3)}\end{aligned}$$

For $m \geq 3$,

$$\begin{aligned}D(y_2,y_3,y_m) &= D(Bx_1,Ax_2,y_m) = D(Ax_2,Bx_1,y_m)\\ &\leq \phi(\text{Max}\{D(Sx_2,Tx_1,y_m),D(Sx_2,Ax_2,y_m),D(Tx_1,Bx_1,y_m),\\ &\qquad \beta D(Sx_2,Bx_1,y_m),\beta D(Tx_1,Ax_2,y_m)\})\\ &= \phi(\text{Max}\{D(y_2,y_1,y_m),D(y_2,y_3,y_m),D(y_1,y_2,y_m),\\ &\qquad \beta D(y_2,y_2,y_m),\beta D(y_1,y_3,y_m)\})\\ &= \phi(\text{Max}\{D(y_2,y_1,y_m),\beta D(y_2,y_2,y_m),\beta D(y_1,y_3,y_m)\}) \ldots\ldots \text{(ii)}\end{aligned}$$

Case : Suppose $D(y_2,y_3,y_m) \leq \phi(D(y_1,y_2,y_m))$

Then $D(y_2,y_3,y_m) \leq \phi(\phi(k))$ from (i)

$= \phi^2(k)$

Case : Suppose $D(y_2,y_3,y_m) \leq \phi(\beta D(y_2,y_2,y_m))$

$$\begin{aligned}\beta D(y_2,y_2,y_m) &\leq \beta D(y_2,y_2,y_m)+\beta D(y_2,y_1,y_m)+\beta D(y_2,y_2,y_1)\\ &\leq \beta\phi(k)+\beta\phi(k)+\beta\phi(k) \text{ from (i)}\\ &\leq \phi(k)\end{aligned}$$

$\therefore\ D(y_2,y_3,y_m) \leq \phi^2(k)$

Case : Suppose $D(y_2, y_3, y_m) \le \phi(\beta D(y_1, y_3, y_m))$ ...... (iii)

$$\beta D(y_1,y_3,y_m) \le \beta D(y_1,y_2,y_m) + \beta D(y_1,y_2,y_m) + \beta D(y_1,y_3,y_2)$$
$$\le \beta D(y_2,y_3,y_m)+\beta\phi(k)+\beta\phi(k) \text{ from (i)}$$
$$= 2\beta\phi(k)+\beta D(y_2,y_3,y_m) \quad \text{...... (iv)}$$

$$\therefore D(y_2,y_3,y_m) \le \phi[2\beta\phi(k)+\beta D(y_2,y_3,y_m)]$$
$$\le 2\beta\phi(k)+\beta D(y_2,y_3,y_m) \text{ since } \phi(t)<t \text{ for } t>0.$$

$$\therefore D(y_2,y_3,y_m) < \frac{2\beta}{1-\beta}\phi(k)$$

Now (iv) becomes

$$\beta D(y_1,y_3,y_m) \le 2\beta\phi(k) + \frac{2\beta^2}{1-\beta}\phi(k) = \frac{2\beta}{1-\beta}\phi(k)$$

Hence (iii) becomes $D(y_2,y_3,y_m) \le \phi\left[\frac{2\beta}{1-\beta}\phi(k)\right] \le \phi^2(k)$

Thus in all three cases we have $D(y_2,y_3,y_m) \le \phi^2(k)$.

In general for $m \ge n+1$ we have $D(y_n, y_{n+1}, y_m) \le \phi^n(k)$

Now from Lemma 1 (Lemma 2.2 of 2.2[2]) it follows that $\{y_n\}$ is a *D*-Cauchy sequence in *X*. Suppose that $\{y_n\}$ converges to some $u \in X$ and A, B, S, T are (G, H_1, H_2)- orbitally lower semi continuous at *u*.

Then

$$\text{Min}\,\{D(Au,Au,Su), D(Au,Su,Su), D(Bu,Bu,Tu), D(Bu,Tu,Tu)\}$$
$$\le \lim_{n\to\infty} \max\{\max\{D(Ax_{2n},Ax_{2n},Sx_{2n}), D(Ax_{2n},Sx_{2n},Sx_{2n})\},$$
$$\max\{D(Bx_{2n+1},Bx_{2n+1},Tx_{2n+1}), D(Bx_{2n+1},Tx_{2n+1},Tx_{2n+1})\}\}$$
$$= \lim_{n\to\infty} \max\{\max\{D(y_{2n+1},y_{2n+1},y_{2n}), D(y_{2n+1},y_{2n},y_{2n})\},$$
$$\max\{D(y_{2n+2},y_{2n+2},y_{2n+1}), D(y_{2n+2},y_{2n+1},y_{2n+1})\}\} = 0$$

since $\{y_n\}$ is *D*-Cauchy

$\therefore Au = Su$ or $Bu = Tu$

The rest follows from Lemma 4.

Theorem 7: *Let A, B, S and T be four self maps on a D-metric space (X, D) such that*

(7.1) $D(Ax,By,z) \le \phi(\text{Max}\,\{D(Sx,Ty,z), D(Sx,Ax,z), D(Ty,By,z), \beta D(Sx,By,z), \beta D(Ty,Ax,z)\})$

$\forall x,y \in X, z = x$ or y or Az_1 or Bz_2 for some $z_1, z_2 \in X$ where $0 \le \beta \le 1/3$,

$\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is non decreasing, $\phi(t) < t$ for $t > 0$ and $\sum_{n=1} \phi^n(t) < \infty$ for each $t \in T^+$.

(7.2) Assume that for some $x_0 \in X$, there exist sequence $\{x_n\}$ and $\{y_n\}$ such that $y_0 = Sx_0, y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+2} = Sx_{2n+2}$, $n = 0, 1, 2$ ... and the set $\{D(y_a, y_b, y_m)/0 \le a \le 1, 1 \le b \le 2, m \ge 2\}$ is bounded by k.

Then $\{y_n\}$ is a D-Cauchy sequence.

Further assume that

(7.3) $\{y_n\}$ converges to some $u \in X$

(7.4) A, B, S, T are (G^*, H_1, H_2)- orbitally lower semi continuous at u.

Then A, B, S and T have a unique common fixed point.

Proof:- The first part of the proof follows as in Theorem 6.

The rest follows from (7.4) and Lemma 5.

REFEREANCES

[1] B. C. Dhage : *Generalised metric spaces and mappings with fixed point*, Bull. Cal. Math. Soc. 84(4)m (1992), 329-336.

[2] B. C. Dhage : *Some results on common fixed points* –I, Indian J. Pure. Appl. Math 30(8), (1999), 827-837.

[3] B. C. Dhage : Smrati Arya and Jeong Sheok Ume: *A general lemma for fixed point theorems involving more than two maps in D-metric spaces with applications*, Internat. J. Math. Math. Sci. No. 11, Vol. 2003, 661-672.

[4] S. V. R. Naidu, K.P.R. Rao, and N. Srinivasa Rao : *On the topology of D-metric spaces and generation of D-metric spaces from metric spaces*, Internat. J. Math. Math. Sci. (to appear).

[1,3] Department of Applied Mathamatics,
A.N.U.P. G Centre,
NUZVID-521201,
Krishna District,
A.P. (INDIA)
E-mail: kprrao2004@yahoo.com

[2] K. JHA,
Department of Mathematical Sciences
School of Science,
KATHMANDU UNIVERSITY,
P.O. BOX NO. 6250.
Kathmandu, Nepal.
E-mail : jhaknh@yahoo.co.in