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New subclass of univalent function defined by using generalized Salagean operator

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Abstract:

In this paper, we have introduced and studied a new subclass $TD_{\lambda}(\alpha,\beta,\xi;n)$ of univalent functions defined by using generalized Salagean operator in the unit disk $U=\{z:|z|<1\}$. We have obtained among others results like, coefficient inequalities, distortion theorem, extreme points, neighbourhood and Hadamard product properties.

Key Words

Univalent function, Distortion theorem, Neighbourhood, Hadamard product 2000 AMS subject classification. 30C45.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

In [4], Al-oboudi defined a differential operator as follows, for a function $f(z) \in A$,

$$\begin{aligned} D^0 f(z) &= f(z), \\ Df(z) = D^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) \\ &= D_\lambda f(z), \quad \lambda \geq 0 \end{aligned} \tag{1.2}$$

in general

$$D^n f(z) = D_\lambda \left(D^{n-1} f(z) \right). \tag{1.3}$$

If $f(z)$ is given by (1.1), then from (1.2) and (1.3) we observe that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda \right]^n a_k z^k \tag{1.4}$$

when $\lambda = 1$, we get Salagean differential operator [7].

Further, let T denote the subclass of A which consists of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.5}$$

A function $f(z)$ belonging to A is in the class $D_\lambda(\alpha, \beta, \xi; n)$, if and only if

$$\left| \frac{\left(D^n f(z) \right)' - 1}{2\xi \left[\left(D^n f(z) \right)' - \alpha \right] - \left[\left(D^n f(z) \right)' - 1 \right]} \right| < \beta \tag{1.6}$$

where $0 \leq \alpha < 1/2\xi$, $0 < \beta \leq 1$, $1/2 \leq \xi \leq 1$, $n \in N \cup \{0\}$, $z \in U$.

Let $TD_\lambda(\alpha,\beta,\xi;n)=T\cap D_\lambda(\alpha,\beta,\xi;n)$ (1.7)

2. MAIN RESULTS

Theorem 2.1. Let $f(z)$ be defined by (1.5). Then $f(z)\in TD_\lambda(\alpha,\beta,\xi;n)$, if and only if

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_k\le 2\beta\xi(1-\alpha) \tag{2.1}$$

$$0\le\alpha<1/2\xi,\ 0<\beta\le 1,\ 1/2\le\xi\le 1,\ n\in N\cup\{0\},\ \lambda\ge 0.$$

Proof. For $|z|=1$, we get

$$\left|\left(D^n f(z)\right)'-1\right|-\beta\left|2\xi\left[\left(D^n f(z)\right)'-\alpha\right]-\left[\left(D^n f(z)\right)'-1\right]\right|$$

$$=\left|-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\,a_k z^{k-1}\right|-\beta\left|2\xi(1-\alpha)-2\xi\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\,a_k z^{k-1}\right.$$

$$\left.+\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\,a_k z^{k-1}\right|$$

$$\le\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_k-2\beta\xi(1-\alpha)\le 0,$$

by hypothesis. Thus by maximum modulus theorem, we have $f(z)\in TD_\lambda(\alpha,\beta,\xi;n)$.

Conversely, suppose that $f(z)\in TD_\lambda(\alpha,\beta,\xi;n)$, hence the condition (1.6) gives us

$$\left|\frac{\left(D^n f(z)\right)'-1}{2\xi\left[\left(D^n f(z)\right)'-\alpha\right]-\left[\left(D^n f(z)\right)'-1\right]}\right|$$

$$= \left| \frac{-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\, a_k z^{k-1}}{2\xi(1-\alpha)-(2\xi-1)\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\, a_k z^{k-1}} \right| < \beta.$$

Since $|\mathrm{Re}(z)| < |z|$ for all z, we obtain

$$\mathrm{Re}\left\{ \frac{\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\, a_k z^{k-1}}{2\xi(1-\alpha)-(2\xi-1)\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\, a_k z^{k-1}} \right\} < \beta.$$

Letting $z \to 1^-$ through real values, we get (2.1). The result is sharp for the function

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]} z^k, \quad k \geq 2.$$

Corollary 2.1. Let $f(z) \in T$ belong to the class $TD_\lambda(\alpha,\beta,\xi;n)$, then

$$a_k \leq \frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}, \quad k \geq 2. \tag{2.2}$$

Theorem 2.2. Let $f(z) \in T$ belong to the class $TD_\lambda(\alpha,\beta,\xi;n)$, then for $|z| \leq r < 1$, we have

$$r - r^2 \frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \leq \left|D^n f(z)\right| \leq r + r^2 \frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \tag{2.3}$$

$$1 - r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \leq \left|\left(D^n f(z)\right)'\right| \leq 1 + r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}. \tag{2.4}$$

The bounds given by (2.3) and (2.4) are sharp.

Proof. By Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]a_k \le 2\beta\xi(1-\alpha)$$

then, we have

$$2(1+\lambda)^{n}\left[1+\beta(2\xi-1)\right]a_k \le \sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]a_k \le 2\beta\xi(1-\alpha),$$

thus,

$$\sum_{k=2}^{\infty}a_k \le \frac{2\beta\xi(1-\alpha)}{2(1+\lambda)^{n}\left[1+\beta(2\xi-1)\right]}.$$

Hence

$$\begin{aligned}\left|D^{n}f(z)\right| &\le |z|+\sum_{k=2}^{\infty}\left|\left[1+(k-1)\lambda\right]^{n}a_k z^k\right| \\ &\le |z|+|z|^2(1+\lambda)^{n}\sum_{k=2}^{\infty}a_k \\ &\le r+r^2(1+\lambda)^{n}\sum_{k=2}^{\infty}a_k \\ &\le r+r^2\frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)},\end{aligned}$$

and

$$\begin{aligned}\left|D^{n}f(z)\right| &\ge |z|-\sum_{k=2}^{\infty}\left|\left[1+(k-1)\lambda\right]^{n}a_k z^k\right| \\ &\ge |z|-|z|^2(1+\lambda)^{n}\sum_{k=2}^{\infty}a_k \\ &\ge r-r^2(1+\lambda)^{n}\sum_{k=2}^{\infty}a_k \\ &\ge r-r^2\frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)},\end{aligned}$$

thus (2.3) is true. Further,

$$\left|\left(D^n f(z)\right)'\right| \le 1+2r(1+\lambda)^n \sum_{k=2}^{\infty} a_k$$
$$\le 1+r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}$$

and

$$\left|\left(D^n f(z)\right)'\right| \ge 1-2r(1+\lambda)^n \sum_{k=2}^{\infty} a_k$$
$$\ge 1-r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}.$$

The result is sharp for the function $f(z)$ defined by

$$f(z)=z-\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}z^2, \qquad z=\pm r.$$

Theorem 2.3. Let $n\in N\cup\{0\}$, $\lambda\ge 0$, $0\le\alpha_1\le\alpha_2<1/2\xi$, $0<\beta\le 1$, $1/2\le\xi\le 1$.

Then $TD_\lambda(\alpha_2,\beta,\xi;n)\subset TD_\lambda(\alpha_1,\beta,\xi;n)$.

Proof. By assumption we have

$$\frac{2\beta\xi(1-\alpha_2)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}\le\frac{2\beta\xi(1-\alpha_1)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}.$$

Thus, $f(z)\in TD_\lambda(\alpha_2,\beta,\xi;n)$ implies that

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n a_k\le\frac{2\beta\xi(1-\alpha_2)}{k\left[1+\beta(2\xi-1)\right]}\le\frac{2\beta\xi(1-\alpha_1)}{k\left[1+\beta(2\xi-1)\right]}$$

then $f(z)\in TD_\lambda(\alpha_1,\beta,\xi;n)$.

Theorem 2.4. The set $TD_\lambda(\alpha,\beta,\xi;n)$ is the convex set.

Proof. Let $f_i(z)=z-\sum_{k=2}^{\infty}a_{k,i}z^k$ $(i=1,2)$ belong to $TD_\lambda(\alpha,\beta,\xi;n)$ and

let $g(z)=\zeta_1 f_1(z)+\zeta_2 f_2(z)$, with ζ_1 and ζ_2 non negative and $\zeta_1+\zeta_2=1$,

We can write

$$g(z)=z-\sum_{k=2}^{\infty}\left(\zeta_1 a_{k,1}+\zeta_2 a_{k,2}\right)z^k.$$

It is sufficient to show that $g(z)\in TD_\lambda(\alpha,\beta,\xi;n)$ that means

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]\left(\zeta_1 a_{k,1}+\zeta_2 a_{k,2}\right)$$

$$=\zeta_1\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_{k,1}+\zeta_2\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_{k,2}$$

$$\le\zeta_1\left(2\beta\xi(1-\alpha)\right)+\zeta_2\left(2\beta\xi(1-\alpha)\right)=(\zeta_1+\zeta_2)\left(2\beta\xi(1-\alpha)\right)=2\beta\xi(1-\alpha).$$

Thus $g(z)\in TD_\lambda(\alpha,\beta,\xi;n)$.

We shall now present a result on extreme points in the following theorem.

Theorem 2.5. Let $f_1(z)=z$ and

$$f_k(z)=z-\frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}z^k$$

for all $k\ge 2$, $n\in N\cup\{0\}$, $\lambda\ge 0$, $0\le\alpha<1/2\xi$, $0<\beta\le 1$, $1/2\le\xi\le 1$.

Then $f(z)$ is in the subclass $TD_\lambda(\alpha,\beta,\xi;n)$, if and only if it can be expressed in the form $f(z)=\sum_{k=2}^{\infty}\gamma_k z^k$ where $\gamma_k\ge 0$ and $\sum_{k=2}^{\infty}\gamma_k=1$ or $1=\gamma_1+\sum_{k=2}^{\infty}\gamma_k$.

Proof. Let $f(z)=\sum_{k=2}^{\infty}\gamma_k z^k$ where $\gamma_k \geq 0$ and $\sum_{k=2}^{\infty}\gamma_k = 1$. Thus

$$f(z)=z-\sum_{k=2}^{\infty}\frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}\gamma_k z^k$$

and we obtain

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha)}\gamma_k \times \frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}$$

$$=\sum_{k=2}^{\infty}\gamma_k = 1-\gamma_1 \leq 1.$$

In view of Theorem (2.1), this show that $f(z)\in TD_\lambda(\alpha,\beta,\xi;n)$.

Conversely, suppose that $f(z)$ of the form (1.5) belong to $TD_\lambda(\alpha,\beta,\xi;n)$ then

$$a_k \leq \frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}, \qquad k\geq 2.$$

Putting

$$\gamma_k = \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha)}$$

and $\gamma_1 = 1-\sum_{k=2}^{\infty}\gamma_k$, then we have $f(z)=\gamma_1 f_1(z)+\sum_{k=2}^{\infty}\gamma_k f_k(z)$.

This completes the proof.

3. NEIGHBOURHOOD AND HADAMARD PRODUCT PROPERTIES

Definition 3.1. [6]. Let $\gamma_k \geq 0$ and $f(z)\in T$ of the form (1.5).
The (k,γ) – neighbourhood of a function $f(z)$ defined by

$$N_{(k,\gamma)}(f)=\left\{g\in T : g(z)=z-\sum_{k=2}^{\infty}b_k z^k \ \ and \ \sum_{k=2}^{\infty}k\left|a_k-b_k\right|\leq\gamma\right\}, \quad (3.1)$$

For the identity function $e(z) = z$, we have

$$N_{(k,\gamma)}(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \ \ and \ \sum_{k=2}^{\infty} k|b_k| \le \gamma \right\}. \tag{3.2}$$

Theorem 3.1. Let $\gamma = \dfrac{2\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}$. Then $TD_\lambda(\alpha,\beta,\xi;n) \subset N_{k,\gamma}(e)$.

Proof. Let $f(z) \in TD_\lambda(\alpha,\beta,\xi;n)$ then we have

$$2(1+\lambda)^n\left[1+\beta(2\xi-1)\right]\sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_k \le 2\beta\xi(1-\alpha),$$

therefore

$$\sum_{k=2}^{\infty} a_k \le \frac{\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}, \tag{3.3}$$

also we have for $|z| < r$

$$|f'(z)| \le 1 + |z|\sum_{k=2}^{\infty} k a_k \le 1 + r\sum_{k=2}^{\infty} k a_k.$$

In view of (3.3), we have

$$|f'(z)| \le 1 + r\frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}.$$

From above inequalities we get

$$\sum_{k=2}^{\infty} k a_k \le \frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]} = \gamma.$$

therefore, $f(z) \in N_{k,\gamma}(e)$.

Definition 3.2. The function $f(z)$ defined by (1.5) is said to be a member of the subclass $TD_\lambda(\alpha,\beta,\xi,\zeta,n)$ if there exits a function $g(z) \in TD_\lambda(\alpha,\beta,\xi;n)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| \le 1-\zeta,\ z \in U,\ 0 \le \zeta < 1.$$

Theorem 3.2. Let $g(z) \in TD_\lambda(\alpha,\beta,\xi;n)$ and

$$\zeta = 1-\frac{\gamma}{2}d(\alpha,\beta,\xi;n). \tag{3.4}$$

Then $N_{k,\gamma}(g) \subset TD_\lambda(\alpha,\beta,\xi,\zeta;n)$ where $n \in N \cup \{0\}$, $\lambda \ge 0$, $0 \le \alpha < 1/2\xi$, $0 < \beta \le 1$, $1/2 \le \xi \le 1$, $0 \le \zeta < 1$ and

$$d(\alpha,\beta,\xi;n) = \frac{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]-\beta\xi(1-\alpha)}.$$

Proof. Let $f(z) \in N_{k,\gamma}(g)$, then by (3.3) we have $\sum_{k=2}^{\infty} k|a_k - b_k| \le \gamma$, then $\sum_{k=2}^{\infty}|a_k - b_k| \le \gamma/2$.

Since $g(z) \in TD_\lambda(\alpha,\beta,\xi;n)$, we have

$$\sum_{k=2}^{\infty} b_k \le \frac{\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]},$$

therefore,

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=2}^{\infty}|a_k-b_k|}{1-\sum_{k=2}^{\infty} b_k}$$

$$\le \frac{\gamma}{2}\left(\frac{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]-\beta\xi(1-\alpha)}\right) = \frac{\gamma}{2}d(\alpha,\beta,\xi;n) = 1-\zeta.$$

Then by definition 3.2, we get $f(z) \in TD_\lambda(\alpha,\beta,\xi,\zeta;n)$.

Theorem 3.3. Let $f(z)$ and $g(z) \in TD_\lambda(\alpha_1,\beta,\xi;n)$ be of the form (1.5) such that

$f(z)=z-\sum_{k=2}^{\infty}a_k z^k$ and $g(z)=z-\sum_{k=2}^{\infty}b_k z^k$, where $a_k, b_k \geq 0$. Then the Hadamard product $h(z)$ defined by $h(z)=z-\sum_{k=2}^{\infty}a_k b_k z^k$ is in the subclass $TD_{\lambda}(\alpha_2,\beta.\xi;n)$ where

$$\alpha_2 \leq \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]-2\beta\xi(1-\alpha_1)^2}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}.$$

Proof. By Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}a_k \leq 1 \tag{3.5}$$

and

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}b_k \leq 1. \tag{3.6}$$

We have only to find the largest α_2 such that

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_2)}a_k b_k \leq 1.$$

Now, by Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n \left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}\sqrt{a_k b_k} \leq 1, \tag{3.7}$$

we need only to show that

$$\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_2)}a_k b_k \leq \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}\sqrt{a_k b_k}$$

equivalently,

$$\sqrt{a_k b_k} \le \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)} \times \frac{2\beta\xi(1-\alpha_2)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}$$

$$\le \frac{1-\alpha_2}{1-\alpha_1}.$$

But from (3.7), we have

$$\sqrt{a_k b_k} \le \frac{2\beta\xi(1-\alpha_1)}{\left[1+(k-1)\lambda\right]^n \left[1+\beta(2\xi-1)\right]}.$$

Consequently, we need to prove that

$$\frac{2\beta\xi(1-\alpha_1)}{\left[1+(k-1)\lambda\right]^n \left[1+\beta(2\xi-1)\right]} \le \frac{1-\alpha_2}{1-\alpha_1}.$$

or equivalently, that

$$\alpha_2 \le \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right] - 2\beta\xi(1-\alpha_1)^2}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}.$$

Theorem 3.4. Let $f(z) \in TD_\lambda(\alpha,\beta,\xi;n)$ be defined by (1.5) and c any real number with $c > -1$ than the function $G(z)$ defined as $G(z) = \frac{c+1}{z^c}\int_0^z s^{c-1} f(s)ds$, $c > -1$, also belongs to $TD_\lambda(\alpha,\beta,\xi;n)$.

Proof. By virtue of $G(z)$ it follows from (1.5) that

$$G(z) = \frac{c+1}{z^c}\int_0^z \left(s^c - \sum_{k=2}^{\infty} a_k s^{k+c-1} \right) ds$$

$$= z - \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k} \right) a_k z^k.$$

But $$\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]}{2\beta\xi(1-\alpha)}\left(\frac{c+1}{c+k}\right)a_k \le 1,$$

Since $\frac{c+1}{c+k}\le 1$ and by Theorem 2.1, so the proof is complete.

Theorem 3.5. Let $f(z)\in TD_\lambda(\alpha,\beta,\xi;n)$ be defined by (1.5) and

$$F_\mu(z)=(1-\mu)z+\mu\int_0^z\frac{f(s)}{s}ds \quad (\mu\ge 0, z\in U).$$

Then $F_\mu(z)$ is also in $TD_\lambda(\alpha,\beta,\xi;n)$ if $0\le\mu\le 2$.

Proof. Let $f(z)$ defined by (1.5) then

$$F_\mu(z)=(1-\mu)z+\mu\int_0^z\left(\frac{s-\sum_{k=2}^{\infty}a_k s^k}{s}\right)ds$$

$$=z-\sum_{k=2}^{\infty}\frac{\mu}{k}a_k z^k.$$

By Theorem 2.1 and since $\left(\frac{\mu}{k}\le 1\right)$ we have

$$\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]}{2\beta\xi(1-\alpha)}\left(\frac{\mu}{k}\right)a_k$$

$$\le\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]}{2\beta\xi(1-\alpha)}\left(\frac{\mu}{2}\right)a_k \le 1,$$

then $F_\mu(z)$ is in $TD_\lambda(\alpha,\beta,\xi;n)$.

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