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On A Class of Bilateral Generating Function For Hermite Polynomials of Two Variables* $H_n(x,y)$

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Abstract: A new class of bilateral generating functions (3) for Hermite polynomials of two variables $H_n(x,y)$ is obtained. Applications of our result are pointed out.

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1. Introduction

Noticing the existence of the following type of generating function of Hermite polynomials of two variables $H_n(x,y)$ (4)

$$(1.1)\qquad e^{2xt-yt^2}\left(1-\frac{yt}{x}\right)^{2\alpha} H_\alpha\left[x\left(1-\frac{yt}{x}\right)^{-1},\ y\left(1-\frac{yt}{x}\right)^{-4}\right]$$

$$=\sum_{n=0}^{\infty}\frac{H_{\alpha+n}(x,y)\,t^n}{n!},\ \text{where } \alpha \text{ is a non negative integer.}$$

We are led to investigate a more general class of generating function by Lie group-theoretic method.

The main result of our investigation is the following theorem :

Theorem 1. *If there exists a linear generating function of the form*

$$(1.2)\qquad G(x,y,t)=\sum_{n=0}^{\infty} a_n\, H_{\alpha+n}(x,y)\, t^n$$

then the following new bilateral generating function (3) will exist.

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$$(1.3)\quad \left(1-\frac{yz}{x}\right)^{2\alpha} \exp\,(2xz-yz^2)\, G\left[x\left(1-\frac{yz}{x}\right)^{-1},\ y\left(1-\frac{yz}{x}\right)^{-4}, tz\left(1-\frac{yz}{x}\right)^{2}\right]$$

$$=\sum_{K=0}^{\infty}\sum_{n=0}^{k}\frac{a_n}{(k-n)!}\ H_{\alpha+k}(x,y)\,t^n z^k \quad \text{with} \left|\frac{yz}{x}\right|<1$$

2. Group Theoretic Discussion

For $H_n(x,y)$ we have the following differential operator

$$(2.1)\qquad B = Z\left(y\frac{\partial}{\partial x}+\frac{4y^2}{x}\frac{\partial}{\partial y}-\frac{2yz}{x}\frac{\partial}{\partial z}+2x\right)\ (4)$$

such that (3)

$$(2.2)\qquad B\left[Hn(x,y)\,z^n\right] = H_{n+1}(x,y)\,z^{n+1}$$

The extended form of the transformation group generated by B is (3)

$$(2.3)\qquad [\exp bB]\ f(x,y,z)$$

$$=\exp\left\{2bxz\left(1-\frac{byz}{2x}\right)\right\}.\ f\left[x\left(1-\frac{byz}{x}\right)^{-1},\ y\left(1-\frac{byz}{x}\right)^{-4}, z\left(1-\frac{byz}{x}\right)^{2}\right]$$

3. Derivation of the generating functions

Let us consider the generating function

$$(3.1)\qquad G(x,y,t)=\sum_{n=0}^{\infty}a_n\ H_{\alpha+n}(x,y)\,t^n$$

Multiplying both sides by t^{α}, we get

$$(3.2)\qquad t^{\alpha}\,G(x,y,t)=\sum_{n=0}^{\infty}a_n\ H_{\alpha+n}(x,y)\,t^{\alpha+n}$$

Replacing t by tz, we get

$$(3.3)\qquad t^{\alpha}z^{\alpha}\,G(x,y,tz)=\sum_{n=0}^{\infty}a_n\ H_{\alpha+n}(x,y)\,z^{\alpha+n}\,t^{\alpha+n}$$

Operating both member of (3.3) by $\exp\,(bB)$

$$\{\exp\,(bB)\}\quad [t^{\alpha}z^{\alpha}G(x,y,tz)] = \{\exp\,(bB)\}\sum_{n=0}^{\infty}a_n\ H_{\alpha+n}(x,y)\,z^{\alpha+n}\,t^{\alpha+n}$$

The left hand side of the last equation becomes

$$t^{\alpha}\, z^{\alpha}\left(1-\frac{byz}{x}\right)^{2\alpha} \exp\left\{2bxz\left(1-\frac{byz}{2x}\right)\right\} G\left[x\left(1-\frac{byz}{x}\right)^{-1}, y\left(1-\frac{byz}{x}\right)^{-4}, tz\left(1-\frac{byz}{x}\right)^{2}\right]$$

On the other hand using (2.2) the right hand side is reduced to

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_n \frac{b^k}{k!} H_{\alpha+n+k}(x,y)\, z^{\alpha+n+k}\, t^{\alpha+n}$$

Hence we get

$$(3.4)\qquad \left(1-\frac{byz}{x}\right)^{2\alpha} \exp\left\{2bxz\left(1-\frac{byz}{2x}\right)\right\} G\left[x\left(1-\frac{byz}{x}\right)^{-1},\ y\left(1-\frac{byz}{x}\right)^{-4}, tz\left(1-\frac{byz}{x}\right)^{2}\right]$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_n \frac{b^k}{k!} H_{\alpha+n+k}(x,y)\, z^{n+k}\, t^{n}$$

Now putting $b = 1$ we get (1.3) using Lemma 10 of Rainville (Page 56, 57)

4. Application

If we consider $a_n = \frac{1}{n!}$ then $G(x, y, t) = \sum_{n=0}^{\infty} \frac{H_{\alpha+n}(x,y)\, t^n}{n!}$

Also from (4) we have

$$(4.1)\qquad G(x,y,t) = e^{2xt-yt^2}\left(1-\frac{yt}{x}\right)^{2\alpha} H_{\alpha}\left[x\left(1-\frac{yt}{x}\right)^{-1}, y\left(1-\frac{yt}{x}\right)^{-4}\right]$$

Again we obtain from (4)

$$(4.2)\qquad G\left[x\left(1-\frac{yz}{x}\right)^{-1}, y\left(1-\frac{yz}{x}\right)^{-4}, tz\left(1-\frac{yz}{x}\right)^{2}\right]$$

$$= e^{2xzt-2ytz^2-yt^2z^2}\left(\frac{x-yz-ytz}{x-yz}\right)^{2\alpha}.$$

$$H_{\alpha}\left[x\left\{1-\frac{yz(1+t)}{x}\right\}^{-1}, y\left\{1-\frac{yz(1+t)}{x}\right\}^{-4}\right]$$

From (1.3) & (4.2) we obtain

$$(4.3)\qquad \exp\left[2xz(1+t) - yz^2(1+t)^2\right]\left\{1-\frac{yz(1+t)}{x}\right\}^{2\alpha}$$

$$H_\alpha\left[x\left\{1-\frac{yz(1+t)}{x}\right\}^{-1}, y\left\{1-\frac{yz(1+t)}{x}\right\}^{-4}\right]$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{1}{n!(k-n)!}\ H_{\alpha+k}\ (x,y)\ t^n\ z^k$$

Now applying the relation $H_n(x,y) = y^{n/2}\ H_n(x/\sqrt{y})$ we get on putting $y = 1$.

(4.4)
$$\exp\,[2xz\,(1+t) - z^2\,(1+t)^2]\ H_\alpha\,[x - z(1+t)]$$
$$=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{1}{n!(k-n)!}\ H_{\alpha+k}\ (x)\ t^n\ z^k.$$

which is a new bilateral generating function for Hermite polynomial of Single variable.

REFERENCES

[1] Dattoli, G.et. al, *Theory of Generalised Hermite Polynomials*, Computer Math. Applic, 28, 4 (1994) 71–83.

[2] Gould. H. W. & Hopper, A.T., *Operational formulas connoted with two generalisations of Hermite polynomials*, Duke Math, J. 29 (1962), 51–63.

[3] Mobride, E.B., (1971), *Obtaining generating function*; Springer verlag, Berlin.

[4] Panja, G.K., & Basu, D.K., *Lie-theoretic Origin of certain generating functions of the Hermite polynomials* $H_n(x,y)$ Bull. Cal. Math. Soc, Vol–94, June 2002, No 3.

[5] Rainville, E.D., (1960), *Special functions*, Macmillan, New York,

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