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Uniform version of Wiener-Tauberian theorem for equicontinuous subsets of subspaces of $L^1(X,\mu)$

C. R. BHATTA

Abstract: The Wiener-Tauberian theorem for **R** says that the closed translation invariant subspace generated by an $f \in L^1(\mathbf{R})$ is $L^1(\mathbf{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we consider Banach subspace of $L^1(X, m)$ and prove the uniform version of the result for $L^1(X)$ and Segal algebra $S(X)$ on hypergroup X, where X is locally compact hypergroup possessing Haar measure m.

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1. Introduction:

Let X be a locally compact Hausdorff topological space. Suppose that there is a continuous map $x \to \tilde{x}$ from X into X such that $(\tilde{x})^{\sim} = x$. Let μ be a regular Borel measure on X such that supp $\mu = X$.

Let $(B, \|\cdot\|_B)$ be a Banach space of functions on X contained in $L^1(X,\mu)$ satisfying $\|\cdot\|_B \geq \|\cdot\|_1$. Suppose that there is a linear isometric map $f \to f^*$ from B into B such that $(f^*)^* = f$. Let there be maps $\sigma, \tau : X \to L(B, B)$ satisfying

(B.1) $\|\sigma(x)\| \leq C$, $\|\tau(x)\| \leq C$ for all $x \in X$ and some $C \geq 1$.

(B.2) there exists $e \in X$ such that $\sigma(e) = I$

For $\phi \in B^*$, the dual space of B, we define $\phi^*(f) = \phi(f^*)$. It is clear that $\phi^* \in B^*$ and $\|\phi^*\| = \|\phi\|$. The maps σ and τ induces σ^* and τ^*: $X \to L(B^*, B^*)$ defined by $\sigma^*(x)\phi(f) = \phi(\sigma(x)f)$ and $\tau^*(x)\phi(f) = \phi((\tau(x)f)$. It is clear that $\|\sigma^*(x)\| \leq C$, $\|\tau^*(x)\| \leq C$ and $\sigma^*(e) = 1$.

For $\phi \in B^*$ and $f \in B$, we define $f \circledcirc \phi$ and $\phi \circledcirc f$ by $f \circledcirc \phi(x) = \phi^*(\sigma(x)f)$ and $f \circledcirc \phi(x) = \phi(\sigma(\tilde{x})f^*)$.

Lemma 1.1. Let B be a Banach subspace of $L^1(X,\mu)$ satisfying (B.1)–(B.2). Suppose that the measure μ satisfy

(M.1.) The function $x \to f \circledcirc \phi(x)$ and $x \to \phi \circledcirc f(x)$ are measurable.

(M.2.) For each $f \in B^*$ and $f, g \in B$, we have

$$\int_X \phi(\sigma(\tilde{x})f)\, g^*(x)\, d\mu(x) = \int_X \phi(\sigma(x)f)\, g(x)\, d\mu(x).$$

(M.3.) For each $f \in B$, $\phi \in B^*$ and $x \in X$, we have

$$(\tau^*(x)\phi)^*(f) = \phi^*(\sigma(\tilde{x})f)$$

(M.4.) For each $\phi \in B^*$; $f, g \in B$ and $x \in X$, we have

$$\int_X \phi^*(\sigma(\tilde{y})g)\, \sigma(x) f(y)\, d\mu(y)$$
$$= \int_X g(y)(\sigma^*(y)\phi)^*(\sigma(x)f)\, d\mu(y).$$

Then, we have, for $f, g \in B$, $\phi \in B^*$ and $x \in X$

(i) $f \circledcirc \phi,\ \phi \circledcirc f \in L^\infty(X, \mu) \subset B^*$

(ii) $(f \circledcirc \phi)^* = \phi^* \circledcirc f^*$

(iii) $f \circledcirc (\tau^*(x)\phi)(e) = f \circledcirc \phi(\tilde{x})$

(iii) $f \circledcirc (g \circledcirc \phi)(x) = \int_X g(y)(f \circledcirc \sigma^*(y)\phi)(x)\, d\mu(y)$

Proof: The proof of (i) and (ii) are same as in ([3], Lemma 3.1). For (iii)

$$(f \circledcirc (\tau^*(x)\phi))(e) = (\tau^*(x)\phi)^*(\sigma(e)f)$$
$$= (\tau^*(x)\phi)^*(f) = \phi^*(\sigma(\tilde{x})f) \quad \text{(using M.3)}$$
$$= f \circledcirc \phi(\tilde{x}).$$

For (iv)

$$f \circledcirc (g \circledcirc \phi)(x) = \phi^* \circledcirc g^*(\sigma(x)f) = \int_X \phi^* \circledcirc g^*(y)(\sigma(x)f)(y)\, d\mu(y)$$
$$= \int_X \phi^*(\sigma(\tilde{y})g)(\sigma(x)f)(y)\, d\mu(y)$$
$$= \int_X g(y)(\sigma^*(y)\phi)^*(\sigma(x)f)\, d\mu(y)$$
$$= \int_X g(y)(f \circledcirc \sigma^*(y)\phi)(x)\, d\mu(y). \qquad \square$$

Theorem 1.2. *Let X be a separable locally compact Hausdorff topological space. Suppose that B is a Banach space of functions on X satisfying* (B.1.)–(B.2.) *and μ a measure satisfying* (M.1.)–(M.4.). *Let $\mathcal{H} \subset B$ be such that $\{\Phi_h : h \in \mathcal{H}\}$ is uniformly equicontinuous. Suppose that there exists $h_0 \in S_1^B$ such that $|h(t)| \le |h_0(t)|$ and $\|h\|_B \le \|h_0\|_B$ for all $h \in \mathcal{H}$ and $t \in X$. Let $\mathcal{U} \subset S_1^{B^*}$ be such that $\tau^*(x)\phi \in \mathcal{U}$ for all $x \in X$ and $\phi \in \mathcal{U}$. If $g \in S_1^B \cap U$ and for any $x, y \in X$, $g \circledcirc \sigma^*(x)\tau^*(y)\phi$ vanishes at infinity for ϕ in $\mathcal{U}$ then $h \circledcirc \phi$ vanishes at infinity for ϕ in $\mathcal{U}$ and h in $\mathcal{H}$.*

Proof: Assume to the contrary that there exists $\delta > 0$ such that for every compact set K in X there exists $x_K \in X \sim K$, $h_K \in \mathcal{H}$ and $\phi_K \in \mathcal{U}$ satisfying $|h_K \circledcirc \phi_K(x_K)| > \delta$.

Since X is separable and locally compact so X is σ-compact. Thus there exists an increasing sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact set with $K_n \subset \operatorname{int} K_{n+1}$ and for F any compact subset in X there exists n_0 with $F \subset K_{n_0}$. Write $h_{K_n} = h_n$, $\phi_{K_n} = \phi_n$ and $x_{K_n} = x_n$. We define a sequence of functions on X by

$$s_n(x) = (h_n \circledcirc \tau^*(\widetilde{x}_n)\,\phi_n)(x)$$
$$\begin{aligned} |s_n(x)| &= |(\tau^*(\widetilde{x}_n)\phi_n)^*(\sigma(x)h_n| \\ &\leq \|(\tau^*(\widetilde{x}_n)\phi_n)^*\|_{B^*}\,\|(\sigma(x)h_n\|_B \\ &\leq C^2\|\phi_n\|_{B^*}\,\|h_n\|_B \leq C^2 \end{aligned}$$

Therefore $s_n \in L^\infty \subset B^*$.

Since $x \to \Phi_h(x)$ is uniformly equicontinuous so for given $\in > 0$ there exists a neighbourhood U_x of x in X such that for $y \in U_x$

$$\|\Phi_h(x) - \Phi_h(y)\|_B < \in/C \text{ for all } h \in \mathcal{H}.$$

Thus for $y \in U_x$, we have

$$\begin{aligned} |s_n(x) - s_n(y)| &= |(\tau^*(\widetilde{x}_n)\,\phi_n)^*(\sigma(x)h_n - (\sigma(y)h_n)| \\ &\leq \|(\tau^*(\widetilde{x}_n)\,\phi_n\|_{B^*}\,\|(\sigma(x)h_n - (\sigma(y)h_n)\|_B \\ &\leq C\|\phi_n\|_{B^*}\,\|\Phi_{h_n}(x)\| - \Phi_{h_n}(y)\|_B \\ &\leq C\|\Phi_{h_n}(x) - \Phi_{h_n}(y\|_B < \in. \end{aligned}$$

By Ascoli's theorem ([2], Theorem 1.3.2) there exists a pointwise convergent subsequence $\{Sn_j\}$ converging to a continuous function s on X. Thus for fixed x,y in X

$$(\sigma(x)g)^*(y)\,s_{n_j}(y) \to (\sigma(x)g)^*(y)\,s(y) \text{ as } j \to \infty$$

also $$|(\sigma(x)g)^*(y)\,s_{n_j}(y)| \leq C^2|(\sigma(x)g)^*(y)|$$

and $(\sigma(x)g)^* \in B \subset L^1(X, \mu)$, so by Lebesgue dominated convergence theorem

$$\int_X (\sigma(x)g)^*(y)\,s_{n_j}(y)\,d\mu(y) \to \int_X (\sigma(x)g)^*(y)\,s(y)\,d\mu(y) \text{ as } j \to \infty$$
$$\Rightarrow \quad g \circledcirc s_{n_j}(x) \to g \circledcirc s(x) \text{ as } j \to \infty.$$

But
$$\begin{aligned} g \circledcirc s_{n_j}(x) &= (g \circledcirc (h_{n_j} \circledcirc \tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}))(x) \\ &= \int_X h_{n_j}(y)\,g \circledcirc \sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}(x)\,d\mu(y) \\ &= \int_X V_{n_j}^x(y)\,d\mu(y) \quad \text{(using Lemma 1.1(iv))} \end{aligned}$$

Since $g \circledcirc \sigma^*(y)\,\tau^*(x)\,\phi$ vanishes at infinity uniformly for $\phi \in \mathcal{U}$ so there exists a compact set K_k such that

$$|g \circledcirc \sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}(x)| < \tfrac{1}{k}$$

whenever $x \notin K_k$

$$\begin{aligned} |V_{n_j}^x(y)| &\leq |h_0(y)|\,|g \circledcirc \sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}(x)| \\ &\leq |h_0(y)|\,\|\sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}\|_B * \|\sigma(x)g\|_B \\ &\leq C^3|h_0(y)|. \end{aligned}$$

Applying Lebesgue dominated convergence theorem

$$\int_X V_{n_j}^x(y)\,d\mu(y) \to 0 \text{ as } j \to \infty.$$

$$\Rightarrow g \odot s_{n_j}(x) \to 0 \text{ as } j \to \infty.$$

so $g \odot s = 0$ but $g \in U$ so $s = 0$.

But $$s_n(e) = n_0 \odot \tau^*(\tilde{x}_n)\,\phi_n(e)$$

$$= (\tau^*(\tilde{x}_{n_j})\,\phi_n)^*(h_n)$$

$$= \phi_n^*\,(\sigma(x_n)h_n) = h_n \odot \phi_n(x_n)$$

so $|s_n(e)| > \delta$. Thus $|s(e)| \geq \delta$ which is a contradiction. □

We now note that (B.1) – (B.2) and (M.1) – (M.4) above are satisfied if X is a locally compact hypergroup possessing a left Haar measure μ (in particular, if X is a locally compact group) and $B = L^1(X, \mu)$. In this case $\sigma(x)f = {}_xf$.

Definition 1. 3 : A hypergroup is a locally compact space X and a binary mapping $(x,y) \to p_x * p_y$ of $X \times X$ into $M(X)$ satisfying the following:

(i) The mapping $(x,y) \to p_x * p_y$ extends to a bilinear associative operation $*$ from $M(X) \times M(X)$ into $M(X)$ such that

$$\int_X d\mu * \gamma = \int_X \int_X \int_X f\,d(p_x * p_y)\,d\mu(x)\,d\gamma(y) \text{ for all } f \in C_0(x).$$

(ii) For each $x, y \in X$, the measure $p_x * p_y$ is a probability measure with compact support.

(iii) The mapping $(\mu, \gamma) \to \mu * \gamma$ is continuous from $M^+(X) \times M^+(X)$ into $M^+(X)$ where $M^+(X)$ is given the weak topology with respect to the family $C_{00}^+(X) \cup \{1\}$.

(iv) There exists an element e in X such that $p_x * p_e = p_e * p_x$ for all $x \in X$.

(v) There exists a homeomorphic involution $x \to \tilde{x}$ of X onto X so that given $x, y \in X$, we have $e \in \operatorname{supp}(p_x * p_y)$ if and only if $y = \tilde{x}$ and $(p_x * p_y)^{\sim} = p_{\tilde{y}} * p_{\tilde{x}}$.

(vi) The map $(x,y) \to \operatorname{supp}(p_x * p_y)$ is continuous from $X \times X$ into the space $C(X)$ of compact subset of X, where $C(X)$ is given the topology studied by Michael, a sub basis for which is given by all $C_{U,V} = \{A \in C(X) : A \cap U \neq \phi \text{ and } A \subset V\}$ where U, V are open subsets of X.

We now note that (B.1) – (B.2) and (M.1) – (M.4) above are satisfied if X is a locally compact hypergroup possessing a left Haar measure μ (in particular, if X is a locally compact group) and $B = L^1(X, \mu)$. In this case $\sigma(x)f = {}_xf$.

Since $\|{}_xf\|_1 \leq \|f\|$ so $\|\sigma(x)\| \leq 1$ for all $x \in X$. The map τ on X is given by $\tau(y)f = \Delta(y)f_y$. For $f \in B$.

$$\|\tau(y)f\|_1 \le \Delta(y) \int_X |f|(x*y)\,d\mu(x)$$

$$= \Delta(y)\Delta(\widetilde{y}) \int_X |f|(x)\,d\mu(x)$$

$$= \|f\|_1 \text{ ([1], 5.3B)}$$

Thus $\|\tau(y)\| \le 1$ for all $y \in X$.

For $\quad f \in B, f^*(x) = \dfrac{f(\widetilde{x})}{\Delta(x)}$.

For $\phi \in B^* = L^\infty(X, \mu)$ and $f \in B$,

$$\phi^*(f) = \int_X \phi(x) f^*(x)\,d\mu(x) = \int_X \frac{\phi(x) f(\widetilde{x})}{\Delta(x)}\,d\mu(x)$$

$$= \int_X \frac{\phi(\widetilde{x}) f(x)}{\Delta(x)\Delta(\widetilde{x})}\,d\mu(x)$$

$$= \int_X \phi(\widetilde{x}) f(x)\,d\mu(x)$$

Thus $\phi^*(x) = \phi(\widetilde{x})$

$$\sigma^*(x)\,\phi = {}_{\widetilde{x}}\phi \text{ and } \tau^*(y)\phi = \phi_{\widetilde{y}}.$$

$$f \odot \phi(x) = \int_X \phi^*(y)\, {}_x f(y)\,d\mu(y)$$

$$= \int_X \phi(\widetilde{y})\, f(x*y)\,d\mu(y)$$

$$= f * \phi(x)$$

(M.1) – (M.2) are satisfied as in ([3], lemma 3.1). For (M.3), let $f \in B$, $\phi \in B^*$, $x \in X$,

$$(\tau^*(x)\phi)^*(f) = \int_X \phi_{\widetilde{x}}(y) \frac{f(\widetilde{y})}{\Delta(y)}\,d\mu(y)$$

$$= \int_X \frac{\phi(y*\widetilde{x}) f(\widetilde{y})}{\Delta(y)}\,d\mu(y)$$

$$= \int_X \phi(\widetilde{y}*\widetilde{x}) f(y)\,d\mu(y)$$

$$= \int_X \int_X \phi(\widetilde{u}) f(y)\,d\mu(y)\,dp_{x} * p_y(u)$$

$$= \int_X \phi^*(x*y) f(y)\,d\mu(y)$$

$$= \int \phi^*(x*y) f(y)\,d\mu(y)$$

$$= \int_X f(\widetilde{x}*y)\phi^*(y)\,d\mu(y) \quad \text{([1], 5.1 D)}$$

$$= \phi^*(\sigma(\widetilde{x})f)$$

For (M.4), let $\phi \in B^*$, $f, g \in B$ and $x \in X$

$$\begin{aligned}
&\int_X \phi^*(\sigma(\widetilde{y})g)(\sigma(x)f)(y)\,d\mu(y)\\
&= \int_X \int_X \phi(\widetilde{u})g(\widetilde{y}*u)f(x*y)\,d\mu(y)\,d\mu(u)\\
&= \int_X \int_X \phi(\widetilde{u})g(\widetilde{y})\ {}_xf(u*y)\,d\mu(u)\\
&= \int_X g(\widetilde{y}) \int_X \frac{\phi^*(u*\widetilde{y})}{\Delta(y)}\ {}_xf(u)\,d\mu(u)\,d\mu(y)\\
&= \int_X \int_X g(y)\phi^*(u*g)\ {}_xf(u)\,d\mu(y)\,d\mu(u)\\
&= \int_X \int_X g(y)\phi(\widetilde{y}*\widetilde{u})\ {}_xf(u)\,d\mu(y)\,d\mu(u)\\
&= \int_X g(y)(\sigma^*(y)\phi)^*(\sigma(x)f)\,d\mu(y) \qquad \square
\end{aligned}$$

Thus we have the following generalization from separable locally compact group G to separable locally compact hypergroup X ([3], Theorem 2.3)

Theorem 1.4. *Let X be a separable locally compact hypergroup possessing a left Haar measure μ. Let $\mathcal{H} \subset L^1(X)$ be such that the family $\{\Phi_h : h \in \mathcal{H}\}$ is left uniformly equicontinuous. Suppose that there exists $h_0 \in S_1$ such that $|h(t)| \leq |h_0(t)|$ for all $h \in \mathcal{H}$ and $t \in X$. Let $\mathcal{U} \subset S_\infty$ be left translation invariant. If $g \in U_0$ and $g * a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ then $h * a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ and $h \in \mathcal{H}$.*

2. Segal Algebras on Hypergroups

Let X be a locally compact hypergroup possessing a left Haar measure μ. Segal algebras on locally compact hypergroups have studied and defined in [5] and [8] (For Segal algebras on groups see [4].

Definition 2.1. Let $S(X)$ be a subspace of $L^1(X)$ which is a Banach space under a norm $\|\cdot\|_S$ such that $\|\cdot\|_S \geq \|\cdot\|_1$ and

S(i) $S(X)$ is dense in $L^1(X)$.

S(ii) $S(X)$ is left translation invariant and for some $\eta > 0$, $\|{}_xf\|_S \leq \eta \|f\|_S$ For each $f \in S(X)$ and $x \in X$.

S(iii) For each $f \in S(X)$, the mapping $x \to {}_xf$ of X into $S(X)$ is continuous.

Then $S(X)$ will be called a Segal algebra. $S(X)$ is said to be symmetric Segal algebra if for $f \in S(X)$, $f^* \in S(X)$ where

$$f^*(x) = \frac{f(\widetilde{x})}{\Delta(x)} \text{ and } \|f\|_S = \|f^*\|_S$$

In fact $S(X)$ is Banach algebra under convolution. This can be seen as in ([6], § 4) Using vector valued integrals as in ([6], § 11, Lemma 1), the following result follows

Lemma 2.2. For any $\phi \in (S(X))^*$, $f \in L^1(X)$ and $g \in S(X)$, the following hold.

(i) $\phi(f * g) = \int_X f(y)\phi({}_{\tilde{y}}g)\, d\mu(y)$

If $S(X)$ is symmetric, then

(ii) $\phi(g * f) = \int_X f(y)\phi(g_{\tilde{y}})\, d\mu(y)$.

Let $B = S(X)$ be a symmetric Segal algebra. Taking $\sigma(x)f = {}_x f$ and $\tau(y)f = \Delta(y)\, f_y$ we have $\| \sigma(x) \| \le 1$ and $\|\tau(x)\| \le 1$ for all $x \in X$. Note that $({}_{\tilde{y}} f)^* = \Delta(y)\, (f^*)_y$. Since $x \to {}_x f$ is continuous so $x \to f \odot \phi(x)$ and $x \to \phi \odot f(x)$ are measurable. Thus (M.1) is satisfied. For $\phi \in (S(X))^*$, $f, g \in S(X)$, we have

$$\int_X \phi({}_{\tilde{x}} f) g^*(x)\, d\mu(x)$$

$$= \int_X \phi({}_x f) \frac{g^*(\tilde{x})}{\Delta(x)}\, d\mu(x) = \int_X \phi({}_x f)\, g(x)\, d\mu(x)$$

Thus (M.2) is satisfied

$$((\tau^*(x)\phi)^*(f) = \phi(\tau(x) f^*) = \phi(\Delta(x)(f^*)_x$$
$$= \phi(({}_{\tilde{x}} f)^*) = \phi^*(\sigma(\tilde{x}) f)$$

For (M.4), let $\phi \in S(X)^*$, $f, g \in S(X)$ and $x \in X$ we have

$$\int_X \phi^*(\sigma(\tilde{y}) g) \sigma(x) f(y)\, d\mu(y)$$
$$= \int_X \phi^*(({}_{\tilde{y}} g))\, {}_x f(y)\, d\mu(y).$$
$$= \phi(g^* * ({}_x f)^*)$$
$$= \phi^*({}_x f * g)$$
$$= \int_X g^*(y) \phi({}_{\tilde{y}}({}_x f)^*)\, d\mu(y)$$
$$= \int_X g(y) \phi({}_y({}_x f)^*)\, d\mu(y).$$
$$= \int_X g(y) (\sigma^*(y)\phi)^*(\sigma(x) f)\, d\mu(y). \qquad \square$$

Thus (M.4) is satisfied. Hence we have the following uniform version of the Wiener Tauberian Theorem for Segal algebras.

Theorem 2.3. *Let X be a locally compact hypergroup possessing a left Haar measure μ. Suppose that $S(X)$ is a symmetric Segal algebra on X. Let $\mathcal{H} \subset S(X)$ be such that the family $\{\Phi_h : h \in \mathcal{H}\}$ is left uniformly equicontinuous. Suppose that there exists $h_0 \in S_1$ (unit ball in $S(X)$) such that $|h(t)| \le |h_0(t)|$ and $\|h\|_S \le \|h_0\|_S$ for all $h \in \mathcal{H}$ and $t \in X$. Let $\mathcal{U} \subset S_\infty$ (unit ball in $S(X)^*$) be such that $\sigma^*(x) \in \mathcal{U}$ for all $\phi \in \mathcal{U}$. If $g \in U_0$ and $g \odot a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ then $h \odot a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ and $h \in \mathcal{H}$.*

3. Examples :

Let X be a unimodular locally compact hypergroup possessing a left Haar measure μ.

(a) $S(X) = L^1(X) \cap L^p(X)\ (1 \le p < \infty)$

$\|f\|_S = \|f\|_1 + \|f\|_p$

Then $S(X)$ is a Segal algebra

S(i) follows since $C_\infty(X)$ is dense in $S(X)$

S(ii) follows from ([1], 3.3 B)

S(iii) follows from ([1], 5.4, 2.2 B)

Clearly $S(X)$ is symmetric

(b) $S(X) = L^1(X) \cap C_0(X)$

$\| f \|_S = \| f \|_1 + \| f \|_\infty,\ f \in S(X)$

Then $S(X)$ is a Segal algebra

S(i) follows since $C_\infty(X)$ is dense in $S(X)$

S(ii) follows from ([1], 3.3 B)

S(iii) follows from ([1], 2.2B, 4.2F)

Note that $S(X)$ is symmetric since $\|f^*\|_\infty = \|f\|_\infty$.

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CHET RAJ BHATTA

Central Department of Mathematics,
Tribhuvan University,
Kirtipur, Kathmandu, Nepal.

E-mail: crbhatta@yahoo.com